A NEW TYPE OF VECTOR FIELD AND INVARIANT DIFFERENTIAL SYSTEMS

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In [1] Robert Hermann introduced the concept of tangent vector fields on the space of maps of one manifold into another. A special type of these are the " k -vector fields" which were studied in [3], where this author defined their bracket and exponential. This paper explores further the analogy with classical continuous groups. Specifically, we study invariance of systems of partial differential equations under k -vector fields.

1. Introduction. Every map and manifold is C^∞ unless otherwise noted. $J^k = J^k(N, M)$ is the manifold of k -jets $j_x^k(f)$ of order k of maps $f: N \rightarrow M$ from the manifold N to the manifold M . α and β are the source and target projections, $\rho_i^{k+i}: J^{k+i} \rightarrow J^i$ the usual projection. $T(M)$ denotes the tangent bundle to M , M_y the tangent space at $y \in M$, π the tangent bundle projection. $C^\infty(Q)$ is the algebra (over the reals R) of C^∞ real-valued functions on the manifold Q .

A k -vector field is a map $\theta: C^\infty(M) \rightarrow C^\infty(J^k)$ which is linear over R and satisfies

$$\theta(FG) = (F \circ \beta)\theta(G) + (G \circ \beta)\theta(F).$$

In [3] the i th prolongation $P^i\theta: C^\infty(J^i) \rightarrow C^\infty(J^{i+k})$ was defined. This satisfies, for $H \in C^\infty(M)$, F and $G \in C^\infty(J^i)$, $P^i\theta(FG) = (F \circ \rho_i^{k+i})P^i\theta(G) + (G \circ \rho_i^{k+i})P^i\theta(F)$ and $P^i\theta(H \circ \beta) = \theta(H) \circ \rho_i^{k+i}$. Using these facts one sees that if θ and ψ are k - and i -vector fields, respectively, then $[\theta, \psi] = P^i\theta \circ \psi - P^k\psi \circ \theta$ is a $(k+i)$ -vector field.

In local coordinates (x^i) on N , (y^λ) on M , $(x^i, y^\lambda, p_{j_1}^\lambda, \dots, p_{j_1 \dots j_k}^\lambda)$ on J^k , where $i, j_1, \dots, j_k = 1, \dots, n$; $\lambda = 1, \dots, m$, we follow Kuranishi [4] in defining for each $F \in C^\infty(J^k)$, $\partial_j^\# F \in C^\infty(J^{k+1})$ by

$$\partial_j^\# F = \frac{\partial F}{\partial x^j} + \frac{\partial F}{\partial y^\lambda} p_j^\lambda + \dots + \frac{\partial F}{\partial p_{j_1 \dots j_k}^\lambda} p_{j_1 \dots j_k}^\lambda.$$

If θ is a k -vector field, then in local coordinates θ may be expressed in the form $\theta = a^\lambda(\partial/\partial y^\lambda)$ where the a^λ are real-valued functions on the coordinate neighborhood U in J^k . If $F \in C^\infty(M)$, then $\theta(F)$ on U is

Received by the editors January 21, 1963 and, in revised form, May 25, 1963.

¹ Supported by ONR Contract 202736.

the function $a^\lambda(\partial F/\partial y^\lambda \circ \beta)$. In these local coordinates the i th prolongation of θ has the expression

$$P^i\theta = a^\lambda \frac{\partial}{\partial y^\lambda} + \partial_j^\# a^\lambda \frac{\partial}{\partial p_j^\lambda} + \cdots + \partial_{j_1}^\# \cdots \partial_{j_i}^\# a^\lambda \frac{\partial}{\partial p_{j_1 \dots j_i}^\lambda}.$$

(See [3, Lemma 1].) We shall also need the following lemma whose proof we omit.

LEMMA 1. *Let θ be a k -vector field, $F_1, F_2 \in C^\infty(J^i)$, $G \in C^\infty(M)$ and $F \in C^\infty(J^{i-j})$, where $0 < j < i$. Then*

$$(A) \quad P^i\theta(F \circ \rho_{i-j}^i) = (P^{i-j}\theta(F)) \circ \rho_{i-j+k}^{i+k},$$

$$(B) \quad P^i\theta(G \circ \beta) = \theta(G) \circ \rho_i^{i+k},$$

$$(C) \quad P^i\theta(F_1 F_2) = (F_1 \circ \rho_i^{i+k}) P^i\theta(F_2) + (F_2 \circ \rho_i^{i+k}) P^i\theta(F_1),$$

$$(D) \quad P^i\theta(\partial_{j_1}^\# \cdots \partial_{j_r}^\# G \circ \rho_r^i) = \partial_{j_1}^\# \cdots \partial_{j_r}^\# \theta(G) \circ \rho_{r+i}^{i+k}, \quad r < k,$$

(E) $P^i\theta(\partial_{j_1}^\# \cdots \partial_{j_r}^\# F \circ \rho_{i-j+r}^i) = (\partial_{j_1}^\# \cdots \partial_{j_r}^\# P^{i-j}\theta(F)) \circ \rho_{i-j+k+r}^{i+k}, \quad r < j$.
Conversely, if $\phi: C^\infty(J^i) \rightarrow C^\infty(J^{i+k})$ satisfies (A), $\cdots$, (E) when $P^i\theta$ is replaced by ϕ , then $\phi = P^i\theta$.

Another important property for us is that if $F \in C^\infty(J^i)$, $f: N \rightarrow M$, then $(\partial/\partial x^i)G(j^i(f)) = (\partial_j^\# G)(j^{i+1}(f))$ for all $G \in C^\infty(J^i)$ [2, Proposition 1.10].

Let $I = (-\epsilon, \epsilon)$. An *integral curve* of θ starting at $f_0: N \rightarrow M$ is a 1-parameter family $f: N \times I \rightarrow M$ with $f_0(x) = f(x, 0)$ and for every $F \in C^\infty(M)$,

$$\theta(j_x^k(f))(F) = \theta(F)(j_x^k(f)) = \frac{\partial}{\partial t} (F \circ f)(x, t) = \left(\left(f * \frac{\partial}{\partial t} \right) F \right) (x, t).$$

2. Differential systems. A system Σ of partial differential equations (s.p.d.e.) of order h with N as independent and M as dependent variables is a finitely generated ideal in $C^\infty(J^h)$. A solution of Σ is a map $f: N \rightarrow M$ such that $F(j_x^h(f)) = 0$ for all $x \in N$, $F \in \Sigma$. $P^k\Sigma$ denotes the s.p.d.e. of order $h+k$ generated by the functions $F \circ \rho_h^{h+k}$, $\partial_j^\# F \circ \rho_{h+1}^{h+k}, \dots, \partial_{j_1}^\# \cdots \partial_{j_k}^\# F$, $1 \leq j, j_i \leq n$, $F \in \Sigma$.

DEFINITION. A k -vector field θ leaves Σ *invariant* if for each $F \in \Sigma$, $P^k\theta(F) \in P^k\Sigma$.

Compare with [2] for the older theory. The intuitive meaning of invariance under a transformation group was that the transformations permute the solutions. We shall show that if f_0 is a solution of Σ which belongs to an integral curve of θ , then Σ evaluated at this integral curve has zero derivatives at f_0 of all orders.

LEMMA 2. *If θ is an invariant vector field of Σ , then θ is an invariant vector field for $P^i\Sigma$, all i .*

This follows from (D) and (E) in Lemma 1. Using local coordinates, a calculation proves

LEMMA 3. If $F \in C^\infty(J^i)$, $f: N \times I \rightarrow M$, and $(\partial f / \partial t) = \theta(j_x^k(f))$, then

$$\frac{\partial}{\partial t} F(j_x^i(f)) = P^i \theta(F) \big|_{j_x^{k+i}(U)}.$$

LEMMA 4. If $f: N \rightarrow M$ is a solution of Σ , it is a solution of $P^i \Sigma$, all i .

THEOREM 1. Suppose that

(A) θ is an invariant k -vector field of Σ ,

(B) $f: N \times I \rightarrow M$ satisfies $(\partial f / \partial t) = \theta(j_x^k(f))$, and

(C) $f(\cdot, 0): N \rightarrow M$ is a solution of Σ .

Then $(\partial^n / \partial t^n) F(j_x^h(f)) \big|_{t=0} = 0$ for all $x \in N$, $F \in \Sigma$, and $n = 1, 2, \dots$.

PROOF. From Lemma 3,

$$\frac{\partial}{\partial t} F(j_x^h(f)) = P^h \theta(F) \big|_{j_x^{k+h}(U)}.$$

However, $P^h \theta(F) \in P^k \Sigma$, and f is a solution of $P^k \Sigma$ by Lemma 4. Hence $P^h \theta(F)(j_x^{k+h}(f)) \big|_{t=0} = 0$, all $x \in N$. Let $F^1 = P^h \theta(F) \in P^k \Sigma$. By Lemma 3,

$$P^{h+k} \theta(F) \big|_{j_x^{2k+h}(U)} = \frac{\partial}{\partial t} F^1(j_x^{h+k}(f)) = \frac{\partial}{\partial t} \left[\frac{\partial}{\partial t} F(j_x^h(f)) \right].$$

Using Lemma 4 as before, $(\partial^2 / \partial t^2) F(j_x^h(f)) \big|_{t=0} = 0$. Continuing in this way, the result follows. Q.E.D.

When the manifolds and functions are real-analytic, Theorem 1 implies that integral curves of an invariant vector field which pass through one solution yield solutions for all parameter values.

3. Lie algebra structure.

PROPOSITION. Let θ and ψ be k - and h -vector fields, respectively. Then

$$P^i[\theta, \psi] = P^{i+h} \theta \circ P^i \psi - P^{i+k} \psi \circ P^i \theta.$$

PROOF. By induction on i . A local coordinate calculation shows the result for $i = 1$. Call $\phi: C^\infty(J^i) \rightarrow C^\infty(J^{i+h+k})$ the operator on the right-hand side. We shall use Lemma 1. Let $F_1, F_2 \in C^\infty(J^i)$, $G \in C^\infty(M)$, and $F \in C^\infty(J^{i-j})$.

$$\begin{aligned} P^{i+h} \theta \circ P^i \psi(F \circ \rho_{i-j}) &= P^{i+h} \theta(P^{i-j} \psi(F) \circ \rho_{i+h-j}) \\ &= (P^{i+h-j} \theta(P^{i-j} \psi(F))) \circ \rho_{i+h+k-j}^{i+h+k} \\ &= (P^{i+h-j} \theta P^{i-j} \psi)(F) \circ \rho_{i+h+k-j}^{i+h+k}, \end{aligned}$$

applying Lemma 1(A) to ψ and θ . Interchanging θ and ψ , we find

$$P^i \phi(F \circ \rho_{i-j}^i) = (P^{i-j} \phi(F)) \circ \rho_{i+h+k-j}^{i+h+k}.$$

Now, by induction, $P^{i-j} \phi(F) = P^{i-j}[\theta, \psi]$. Hence (A) holds for ϕ . The same technique works for (B), $\dots$, (E). Q.E.D.

THEOREM 2. *If θ and ψ are k - and h -vector fields, respectively, which leave Σ invariant, then $[\theta, \psi]$ leaves Σ invariant.*

PROOF. If $F \in \Sigma$ and Σ is of order i , then $P^i[\theta, \psi](F) = P^{i+h}\theta \circ P^i\psi(F) - P^{i+k}\psi \circ P^i\theta(F)$. However, $P^i\psi(F) \in P^h\Sigma$. By Lemma 2, θ is an invariant vector field of $P^h\Sigma$, so $P^{i+h}\theta \circ P^i\psi(F) \in P^{h+k}\Sigma$. Similarly $P^{i+k}\psi \circ P^i\theta(F) \in P^{h+k}\Sigma$. Q.E.D.

We conclude that the set of all k -vector fields, $k=1, 2, \dots$, leaving Σ invariant forms a Lie algebra under the bracket.

4. An example. Let $N = E^n$, $M = E^m$. Consider an s.p.d.e. of the type

$$\frac{\partial y^\lambda}{\partial x^n} = \phi^\lambda \left(x^1, \dots, x^{n-1}, y^\mu, \frac{\partial y^\mu}{\partial x^1}, \dots, \frac{\partial y^\mu}{\partial x^{n-1}} \right),$$

$\lambda, \mu = 1, \dots, m$. On J^1 let $F^\lambda = p_n^\lambda - \phi^\lambda(x^i, y^\mu, p_i^\mu)$, and let Σ be generated by $F^1, \dots, F^m$. Then by a calculation one may check that $\theta = \phi^\lambda(\partial/\partial y^\lambda)$ turns out to be an invariant vector field of Σ .

We can see that θ generates solutions of the Cauchy problem associated with Σ . Since θ is independent of x^n and p_n^λ , it can be considered a 1-vector field on E^{n-1} . Suppose $f_0: E^{n-1} \rightarrow E^m$ is the initial data at $x^n = 0$. Suppose $I = \{x^n \mid -\epsilon < x^n < \epsilon\}$ and $f: E^{n-1} \times I \rightarrow E^m$ is an integral curve of θ through f_0 . But that is merely another way of saying that f is a solution of Σ .

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