A FURTHER EXTENSION OF LEBESGUE'S COVERING THEOREM

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1. **Introduction.** Generalizing Lebesgue's covering theorem [3, Theorem IV. 2], Saadaldin has proven [5, Theorem 1.1] the following theorem:

If X is a compact metric space and $\dim X \geq n$, then there exists an $\epsilon > 0$ such that if $\mathcal{G}$ is a topologically contracting sequence of closed sets covering X such that the diameter of G is less than ϵ for all G in $\mathcal{G}$, then there is a point p in X such that p is common to at least $n+1$ elements of $\mathcal{G}$.

A collection $\mathcal{G}$ of subsets of a topological space X will be called *topologically contracting* provided if $G_1, G_2, \dots$ is a sequence of distinct elements of $\mathcal{G}$ and (a_i) and (b_i) are two sequences of points such that for each k , a_k and b_k are in G_k , then either neither sequence converges or they both converge to the same point. This definition is due to R. L. Moore [4, p. 341]. A countable topologically contracting collection will be called a *contracting sequence*. In this paper Saadaldin's result is generalized to arbitrary metric spaces by omitting the hypothesis of compactness and changing the conclusion to the natural replacement [3, p. 60]. If $\mathcal{G}$ is a collection of subsets of a topological space X , the *order* of $\mathcal{G}$ is $\max \{n: \text{There are } n \text{ distinct elements of } \mathcal{G} \text{ with a point in common}\}$. Thus the main result is the following theorem, where $\dim X$ is the covering dimension of X :

THEOREM 1. *Let X be a metric space. Then $\dim X \geq n$ if and only if there is a finite open cover $\mathcal{U}$ of X such that if $\mathcal{G}$ is a contracting sequence of closed sets covering X and refining $\mathcal{U}$, then the order of $\mathcal{G}$ is at least $n+1$.*

For compact metric spaces Saadaldin's condition is sufficient as well as necessary. The proof follows the lines of the proof we shall give.

Saadaldin has shown [5, Theorem 2.1, Theorem 2.2] that some restriction of the cover $\mathcal{G}$, in addition to countability, is needed.

2. **An auxiliary theorem.** The major step in the proof of necessity in Theorem 1 is the proof of an auxiliary theorem which is a gen-

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eralization of Theorem 3.1 in [5]. The proof is by induction, and it follows quickly from the lemmas and assertions which follow the statement of the theorem.

THEOREM 2. *Let X be a metric space. Let $C_1, C'_1; \dots; C_n, C'_n$ be closed subsets of X with the following properties:*

(a) $C_i \cap C'_i = \emptyset$ for all i .

(b) *If B_i is a closed set separating C_i and C'_i for all i , then $\bigcap_{i=1}^n B_i \neq \emptyset$.*

If $\mathcal{G}$ is a contracting sequence of closed sets covering X and if G does not intersect both C_i and C'_i for all $i, i=1, \dots, n$, and for all $G \in \mathcal{G}$, then $\mathcal{G}$ is of order $\geq n+1$.

Let $C_i, C'_i, i=1, \dots, n$, be as in the hypothesis of Theorem 2 and suppose that there is a contracting sequence $\mathcal{G}$ of closed sets covering X such that for all $G \in \mathcal{G}$ and for all $i, i=1, \dots, n$, G does not intersect both C_i and C'_i but such that no point of X belongs to more than n elements of $\mathcal{G}$. Let $\mathcal{G}'$ be the collection of all nonempty intersections of n elements of $\mathcal{G}$; i.e., a subset T of X is in $\mathcal{G}'$ provided T is not empty and there exist distinct elements $G_{k_1}, \dots, G_{k_n}$ in $\mathcal{G}$ such that $T = \bigcap_{i=1}^n G_{k_i}$. That $\mathcal{G}'$ is a countable collection of closed sets is obvious. Furthermore notice that for $n=1$, $\mathcal{G}' = \mathcal{G}$. Let $\mathcal{G} = \{G_1, G_2, \dots\}$; let $\mathcal{G}' = \{T_1, T_2, \dots\}$.

2.1. ASSERTION. $\mathcal{G}'$ is a contracting sequence of disjoint closed sets.

PROOF. Suppose that A and B are elements of $\mathcal{G}'$ and that $x \in A \cap B$. Suppose $A = \bigcap_{i=1}^n A_i$ where $A_i \in \mathcal{G}$ for all i and $A_i \neq A_j$ if $i \neq j$; suppose $B = \bigcap_{i=1}^n B_i$ where B_i is in $\mathcal{G}$ for all i and $B_i \neq B_j$ if $i \neq j$. Then either x belongs to $2n$ elements of $\mathcal{G}$, or not all of $A_1, \dots, A_n, B_1, \dots, B_n$ are distinct. But since by hypothesis x is in at most n distinct elements of $\mathcal{G}$, as the A_i 's are distinct, and as the B_i 's are distinct, we may conclude that for every i , there is one and only one j such that $A_i = B_j$. Thus $A = B$, proving that the elements of $\mathcal{G}'$ are disjoint.

Let (T_i) be a sequence of distinct elements of $\mathcal{G}'$, and let (p_i) and (q_i) be sequences of elements of X such that p_i and q_i are in T_i for all i . Suppose that (p_i) converges to p and that (q_i) does not converge to p . Then there is a neighborhood S of p such that infinitely many q_i 's are not in S . Let (u_i) be a subsequence of (q_i) which contains only elements not in S . Let (v_i) be the corresponding subsequence of (p_i) . (That is, if $u_i = q_j$, $v_i = p_j$.) For all i , let S_i be some element of $\mathcal{G}'$ which contains both v_i and u_i . Then for all i , S_i is $\bigcap_{j=1}^n K_{ij}$ where for all j , $K_{ij} \in \mathcal{G}$. For all i , let $n_i = \max\{k: G_k = K_{ij} \text{ for some } j\}$. Let $R_i = G_{n_i}$. Because there are only finitely many j 's such that $R_i = R_j$ for any i , we may choose subsequences (v_{i_k}) , (u_{i_k}) , and (R_{i_k}) such that (R_{i_k}) is a sequence of distinct elements of $\mathcal{G}$ and such

that (v_{i_k}) and (u_{i_k}) are sequences of elements of X satisfying $v_{i_k}, u_{i_k} \in R_{i_k}$ for all k . Then (v_{i_k}) converges to p , but (u_{i_k}) does not converge to p , contradicting the hypothesis $\mathfrak{G}$ is a contracting sequence. If (q_i) converges to q and (p_i) does not converge to q , we get a contradiction in the same manner, completing the proof of Assertion 2.1.

LEMMA 1. *Let (X, d) be a metric space; let H and K be disjoint closed sets. Let $\mathfrak{L}$ be a contracting sequence of disjoint closed sets such that no element of $\mathfrak{L}$ intersects $H \cup K$. Then there are open sets U and V containing H and K respectively such that no element of $\mathfrak{L}$ intersects both $\bar{U}$ and $\bar{V}$ and $\bar{U} \cap \bar{V} = \emptyset$.*

If A is a subset of X , let $D(A)$ be the diameter of A . If $x \in H$, let $g(x) = d(x, K)$; if $x \in K$, let $g(x) = d(x, H)$. For all x in $H \cup K$, $g(x) > 0$. For each x in $H \cup K$ let $\mathfrak{L}_x$ be the set of all G in $\mathfrak{L}$ such that $D(G) \geq g(x)/3$.

2.2. ASSERTION. *For each x in $H \cup K$, there is a positive real number $h(x)$ such that $d(x, G) \geq h(x)$ for all G in $\mathfrak{L}_x$.*

PROOF. If there is no such number for some x , then for every positive integer k , the ball $B(k)$ about x of radius $g(x)/6k$ intersects some G_k in $\mathfrak{L}_x$. Let p_k be an element in this intersection. For all k , there is in G_k a point q_k which is not in $B(2)$, for otherwise $D(G_k)$ would be less than $g(x)/3$, a contradiction. Clearly (p_k) converges to x . Since $d(x, G_k) > 0$ for all k , we may pick a subsequence (G_{k_n}) of (G_k) in which no two elements are the same. Then (p_{k_n}) converges to x , but (q_{k_n}) does not, contradicting the hypothesis $\mathfrak{L}$ is contracting. Therefore there is such a number, proving our assertion.

We now prove Lemma 1. For $x \in H \cup K$, let

$$f(x) = \frac{1}{2} \min \left\{ \frac{1}{2} h(x), \frac{1}{3} g(x) \right\},$$

and let B_x be the open ball about x of radius $f(x)$. Let $U = \bigcup_{x \in H} B_x$; let $V = \bigcup_{x \in K} B_x$. The proof of Lemma 1 will be complete as soon as we have proven Assertions 2.3 and 2.4.

2.3. ASSERTION. $\bar{U} \cap \bar{V} = \emptyset$.

PROOF. If x is in $\bar{U} \cap \bar{V}$, let $\delta = \frac{1}{6} \min \{d(x, H), d(x, K)\}$. Notice that x is in neither H nor K , for if x is in $H \cap \bar{V}$, then there exist u in V and w in K such that $d(x, u) < f(x)$ and $d(u, w) < f(w)$. In that case,

$$\begin{aligned} d(x, w) &< f(x) + f(w) \leq 2 \max \{f(x), f(w)\} \\ &\leq 2 \max \left\{ \frac{1}{6} g(x), \frac{1}{6} g(w) \right\} \leq \frac{1}{3} d(x, w), \end{aligned}$$

a contradiction. Similarly x is not in K . Since H and K are closed, δ is

positive. There exist q in U and p in V such that q and p are in the ball about x of radius δ . There exist y in H and z in K such that $q \in B_y$ and p is in B_z . We may assume without loss of generality that $f(y) \leq f(z)$. Then

$$\delta \leq \frac{1}{6}d(x, K) \leq \frac{1}{6}d(x, z) \leq \frac{1}{6}(d(x, p) + d(p, z)) < \frac{1}{6}\delta + \frac{1}{6}f(z);$$

that is, $\delta < f(z)$. Then

$$\begin{aligned} d(z, H) &\leq d(y, z) \leq d(y, q) + d(q, p) + d(p, z) < f(y) + 2\delta + f(z) \\ &< 4f(z) \leq 4(\tfrac{1}{2}[\tfrac{1}{3}g(z)]) = \tfrac{2}{3}d(z, H), \end{aligned}$$

a contradiction.

2.4. ASSERTION. If $G \in \mathcal{L}$, G does not intersect both $\bar{U}$ and $\bar{V}$.

PROOF. Suppose $G \cap \bar{U} \neq \emptyset$ and $G \cap \bar{V} \neq \emptyset$. Let x be in $G \cap \bar{U}$; let $y \in G \cap \bar{V}$; let $\epsilon = \frac{1}{3} \min \{d(x, H), d(y, K)\}$. Then ϵ is positive. There is a p in the intersection of U and the ball about x of radius ϵ , and there is a q in the intersection of V and the ball about y of radius ϵ . There exist u in H such that $p \in B_u$ and v in K such that $q \in B_v$. Suppose $f(u) \leq f(v)$, a matter of notation. Then

$$\epsilon \leq \tfrac{1}{3}d(y, K) \leq \tfrac{1}{3}d(y, v) \leq \tfrac{1}{3}(d(y, q) + d(q, v)) < \tfrac{1}{3}(\epsilon + f(v)),$$

or $\epsilon < \frac{1}{3}f(v)$. We now see that $\epsilon < \frac{1}{2}(\frac{1}{2}[\frac{1}{3}g(v)]) \leq \frac{1}{6}g(v)$. Then

$$\begin{aligned} d(v, y) &\leq d(v, q) + d(q, y) \leq f(v) + \epsilon \\ &\leq \tfrac{1}{4}h(v) + \tfrac{1}{3}d(y, K) \leq \tfrac{1}{4}h(v) + \tfrac{1}{3}d(y, v); \end{aligned}$$

that is, $d(v, y) < \frac{1}{2}h(v)$. From this last inequality we may conclude that $d(v, G) < h(v)$. Therefore $D(G) < g(v)/3$. Now

$$\begin{aligned} d(v, H) &\leq d(v, u) \leq d(u, p) + d(p, x) + d(x, y) + d(y, q) + d(q, v) \\ &\leq f(u) + \epsilon + D(G) + \epsilon + f(v) \leq 2f(v) + 2\epsilon + D(G) \\ &< 2(\tfrac{1}{2}[\tfrac{1}{3}g(v)]) + 2(\tfrac{1}{6}g(v)) + \tfrac{1}{3}g(v) = g(v) = d(v, H), \end{aligned}$$

a contradiction.

2.5. ASSERTION. There is a closed set B_n separating C_n and C'_n such that if $G \in \mathcal{G}'$, $G \cap B_n = \emptyset$.

PROOF. Set $F_0 = C_n$; set $F'_0 = C'_n$. F_0 and F'_0 are disjoint closed sets and no element of $\mathcal{G}'$ intersects both F_0 and F'_0 . In general suppose k is a non-negative integer and F_k and F'_k are disjoint closed sets in X such that no element of $\mathcal{G}'$ intersects both F_k and F'_k . Let D_{k+1} be F_k united with all T_i in $\mathcal{G}'$ which intersect F_k and with the first T_i in $\mathcal{G}'$ which intersects neither F_k nor F'_k (if such a T_i exists); let D'_{k+1} be the union of F'_k and all T_i in $\mathcal{G}'$ which intersect F'_k . Now D_{k+1} and

D'_{k+1} are disjoint and closed as $\mathcal{G}'$ is a contracting sequence of disjoint closed sets. Furthermore if $T \in \mathcal{G}'$ and T is not contained in $D_{k+1} \cup D'_{k+1}$, then $T \cap (D_{k+1} \cup D'_{k+1})$ is empty. By Lemma 1, there are open sets E_{k+1} and E'_{k+1} such that $D_{k+1} \subseteq E_{k+1}$, $D'_{k+1} \subseteq E'_{k+1}$, $\bar{E}_{k+1} \cap \bar{E}'_{k+1} = \emptyset$, and T does not intersect both $\bar{E}_{k+1}$ and $\bar{E}'_{k+1}$ for all $T \in \mathcal{G}'$. Let $F_{k+1} = \bar{E}_{k+1}$; let $F'_{k+1} = \bar{E}'_{k+1}$. Starting with F_0 and F'_0 , we obtain the infinite sequences (D_i) , (D'_i) , (E_i) , (E'_i) , (F_i) , and (F'_i) with the following properties for all positive i :

- (1) E_i and E'_i are open sets; $F_i = \bar{E}_i$ and $F'_i = \bar{E}'_i$.
- (2) $F_{i-1} \subseteq D_i \subseteq E_i \subseteq F_i$ and $F'_{i-1} \subseteq D'_i \subseteq E'_i \subseteq F'_i$.
- (3) $F_i \cap F'_i = \emptyset$.
- (4) $C_n \subseteq D_i$; $C'_n \subseteq D'_i$.
- (5) If $T \in \mathcal{G}'$, then for some i , $T \subseteq D_i$ or $T \subseteq D'_i$.

Setting $U = \bigcup_{i=1}^{\infty} E_i$ and $U' = \bigcup_{i=1}^{\infty} E'_i$, we let $B_n = X - (U \cup U')$. Clearly B_n satisfies the requirements of Assertion 2.1.

We now prove Theorem 2. If $X = \emptyset$, the theorem is trivial; assume $X \neq \emptyset$. The proof is by induction on the positive integer n . If $n = 1$, and if we assume a collection $\mathcal{G}$ which satisfies the hypotheses of Theorem 2, but for which the conclusion fails, then, since $\mathcal{G} = \mathcal{G}'$ and $\mathcal{G}$ is a cover of X , Assertion 2.5 implies that the empty set separates C_1 and C'_1 , contradicting the hypothesis of Theorem 2. The induction step of Theorem 2 is exactly the same as in the proof of Theorem 3.1 of [5], for we notice that Proposition A, p. 42 of [3] is true for metric spaces.

3. Proof of Theorem 1. We first prove necessity. Notice that if $n \leq 0$, the open cover may be taken to be the collection whose only element is X . Since $\dim X \geq n$, there exist [1, p. 11] n pairs of closed sets C_i, C'_i ($i = 1, \dots, n$) with the properties listed in the hypothesis of Theorem 2. Clearly if $\mathcal{G}$ is a contracting sequence of closed sets covering X such that no element G of $\mathcal{G}$ intersects both C_i and C'_i for each i , $i = 1, \dots, n$, then by Theorem 2 the order of $\mathcal{G}$ is at least $n+1$. It suffices therefore to find a finite open cover of X such that no element of the cover intersects both C_i and C'_i for any i .

LEMMA 2. *Let X be a topological space and let $C_1, C'_1; \dots; C_n, C'_n$ be pairs of nonempty closed subsets of X such that $C_i \cap C'_i = \emptyset$ for each i . Then there is a finite open cover $\mathcal{U}$ of X such that for any i , no element of $\mathcal{U}$ intersects both C_i and C'_i .*

PROOF. The proof is inductive. If $n = 1$, let $\mathcal{U}$ equal $\{X - C_1, X - C'_1\}$. Given $C_1, C'_1; \dots; C_n, C'_n$, $n > 1$, then by the induction hypothesis, there is a finite open cover $\mathcal{U}'$ of X such that no element of $\mathcal{U}'$ inter-

sects both C_i and C'_i for $i=1, 2, \dots, n-1$. The open sets $U-C_n$ and $U-C'_n$ are such that neither intersects both C_n and C'_n for each U in $\mathfrak{U}$. Let $\mathfrak{U} = \{U-C_n: U \in \mathfrak{U}'\} \cup \{U-C'_n: U \in \mathfrak{U}'\}$. Then $\mathfrak{U}$ is a suitable finite cover.

The proof of sufficiency follows from the observation that a finite collection of sets is topologically contracting. Let $\mathfrak{U}$ be a finite open cover of X with the property described in Theorem 1. Let $\mathfrak{V}$ be a finite open cover of X refining $\mathfrak{U}$. By [2], [4, Proposition 3] there is a finite closed cover $\mathfrak{G}$ of X refining $\mathfrak{V}$ and such that for each $V \in \mathfrak{V}$ there is a unique $G \in \mathfrak{G}$ such that $G \subseteq V$. Since $\mathfrak{G}$ refines $\mathfrak{U}$, the order of $\mathfrak{G}$ is at least $n+1$. It is immediate that the order of $\mathfrak{V}$ is at least $n+1$. We conclude that $\dim X \geq n$.

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