ON COMMUTATORS OF OPERATORS ON HILBERT SPACE

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1. In this note we first generalize a result of P. R. Halmos [3] concerning commutators of (bounded) operators on Hilbert space. Then we obtain some partial results on a problem of commutators in von Neumann algebras which is closely related to another problem raised by Halmos in [4]. Let $\mathcal{H}$ be any (infinite-dimensional) Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded operators on $\mathcal{H}$. We follow Halmos [3] in calling a subspace $\mathcal{K} \subset \mathcal{H}$ *large* if $\mathcal{K}$ contains infinitely many orthogonal copies of $\mathcal{H} \ominus \mathcal{K}$. Halmos proved in [3] that any operator in $\mathcal{L}(\mathcal{H})$ with a large reducing null space is a commutator (of two bounded operators in $\mathcal{L}(\mathcal{H})$). We generalize this to

THEOREM 1. *Any operator in $\mathcal{L}(\mathcal{H})$ which has a large null space is a commutator.*

The construction involved in the proof of this theorem is a generalization of Halmos' construction in [3], and our construction actually yields a slightly more general result than Theorem 1. This more general result does not admit a nice formulation on nonseparable spaces, but on separable spaces it is easy to describe.

THEOREM 2. *Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$. If this decomposition of $\mathcal{K}$ is used to write every operator $T \in \mathcal{L}(\mathcal{K})$ as a 2×2 operator matrix*

$$T = \begin{pmatrix} A & C \\ B & D \end{pmatrix},$$

where the entries are operators on $\mathcal{H}$, then every operator $T \in \mathcal{L}(\mathcal{K})$ of the form

$$T = \begin{pmatrix} A & C \\ B & 0 \end{pmatrix}, \tag{1}$$

where C is a compact operator, is a commutator.

For separable spaces, Theorem 1 is a special case ($C = 0$) of Theorem 2, and the proof of Theorem 1 for nonseparable spaces is an easy modification of the proof of that theorem for separable spaces. Thus we confine ourselves to proving Theorem 2.

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PROOF OF THEOREM 2. Let $\{E_i\}_{i=1}^{\infty}$ be any countable collection of mutually orthogonal projections $E_i \in \mathcal{L}(\mathcal{K})$ such that the sum of the E_i is the identity operator on $\mathcal{K}$ and such that the range of each E_i is an infinite-dimensional subspace of $\mathcal{K}$. Each E_i gives rise to a projection $F_i \in \mathcal{L}(\mathcal{K})$ defined by

$$F_i = \begin{pmatrix} 0 & 0 \\ 0 & E_i \end{pmatrix}, \qquad i = 1, 2, \cdots,$$

and if $F_0 \in \mathcal{L}(\mathcal{K})$ is defined as

$$F_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

then the F_i, $i = 0, 1, 2, \cdots$, are mutually orthogonal projections on $\mathcal{K}$ whose sum is the identity operator on $\mathcal{K}$. Furthermore, the F_i are mutually equivalent in the sense of Murray-von Neumann in the von Neumann (v.N.) algebra $\mathcal{L}(\mathcal{K})$. Thus the F_i together with the implementing partial isometries form a complete set of matrix units for $\mathcal{L}(\mathcal{K})$, and we can use this set of matrix units to regard $\mathcal{L}(\mathcal{K})$ as an infinite matrix algebra. More precisely, it follows from [2, Proposition 5, p. 27], that $\mathcal{L}(\mathcal{K})$ is unitarily equivalent to the v.N. algebra $\mathbf{A}$ of all $\aleph_0 \times \aleph_0$ matrices with entries from the v.N. algebra $F_0\mathcal{L}(\mathcal{K})F_0 \cong \mathcal{L}(\mathcal{K})$ which act as operators on the Hilbert space $\mathcal{K}_1 = \mathcal{K} \oplus \mathcal{K} \oplus \cdots$. (Recall that $F_0(\mathcal{K}) = \mathcal{K} \oplus 0$.) Thus we can and do work with the infinite matrices of $\mathbf{A}$ instead of the 2×2 matrices of $\mathcal{L}(\mathcal{K})$. It is easy to see that any operator in $\mathcal{L}(\mathcal{K})$ of the form

$$\begin{pmatrix} A & C \\ B & 0 \end{pmatrix}$$

is carried by the isomorphism between $\mathcal{L}(\mathcal{K})$ and $\mathbf{A}$ onto an operator in $\mathbf{A}$ of the form

$$X = \begin{pmatrix} A & C_1 & C_2 & C_3 \cdots \\ B_1 & 0 & 0 & 0 \cdots \\ B_2 & 0 & 0 & \cdot \cdots \\ B_3 & 0 & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \\ \cdot & & & \end{pmatrix}.$$

A matrix calculation shows that formally X is the commutator $X = SR - RS$ where

$$S = \begin{pmatrix} -B_1 & A & C_1 & C_2 & C_3 \cdots \\ -B_2 & 0 & A & C_1 & C_2 \cdots \\ -B_3 & 0 & 0 & A & C_1 \cdots \\ \cdot & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

and

$$R = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot \\ \cdot & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

It is obvious that R represents a bounded operator in $\mathbf{A}$, and thus to complete the proof it suffices to show that if C is compact then the collection $\{E_i\}_{i=1}^{\infty}$ can be chosen in such a way as to ensure that S represents a bounded operator. Since X is a bounded operator and since, obviously, the matrix

$$\begin{pmatrix} 0 & A & 0 & \cdot & & \\ & 0 & A & 0 & \cdot & \\ & & 0 & A & 0 & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \end{pmatrix}$$

represents a bounded operator, it is easy to see that it suffices to show that the collection $\{E_i\}$ can be chosen so that the "Toeplitz" matrix

$$Y = \begin{pmatrix} C_1 & C_2 & C_3 & \cdot & \\ 0 & C_1 & C_2 & \cdot & \\ \cdot & 0 & C_1 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

represents a bounded operator. That this can be done follows from the following sequence of lemmas.

LEMMA 1.1. *If C is a compact operator on the separable Hilbert space $\mathcal{H}$, and $C=UP$ is the polar decomposition of C, then there exists a sequence of mutually orthogonal infinite-dimensional subspaces $\mathcal{K}_1, \mathcal{K}_2, \cdots \subset \mathcal{K}$ such that*

(a) $\sum_{i=1}^{\infty} \oplus \mathcal{K}_i = \mathcal{H}$;

(b) *each $\mathcal{K}_i$ is a reducing subspace for P, so that the linear manifolds $\{C(\mathcal{K}_i)\}$ are mutually orthogonal;*

(c) *for each $i \geqq 2$, $\|P \mid \mathcal{K}_i\| = \|C \mid \mathcal{K}_i\| < 1/2^i$.*

PROOF. If P has an infinite-dimensional null space $\mathfrak{N}$, take $\mathcal{K}_1$ to be the direct sum of $\mathcal{H} \ominus \mathfrak{N}$ and a sufficiently large subspace $\mathfrak{M} \subset \mathfrak{N}$ to ensure that $\mathcal{K}_1$ is infinite-dimensional. Arrange it so that $\mathfrak{N} \ominus \mathfrak{M}$ remains infinite-dimensional, and write $\mathfrak{N} \ominus \mathfrak{M} = \mathcal{K}_2 \oplus \mathcal{K}_3 \oplus \cdots$, where $\mathcal{K}_2, \mathcal{K}_3, \cdots$ are also infinite-dimensional. If $\mathfrak{N}$ is finite-dimensional, then

$$P = \sum_{i=1}^{\infty} \alpha_i E_i,$$

where each α_i is a positive scalar, $\{\alpha_i\} \to 0$, and the E_i are mutually orthogonal projections on finite-dimensional spaces. Now the problem is essentially that of partitioning a countable set into a countable union of disjoint infinite subsets, maintaining some care so that (c) will be satisfied. We omit further details of that argument.

LEMMA 1.2. *With C, $\mathcal{H}$ and the sequence $\{\mathcal{K}_i\}$ as in Lemma 1.1, for each positive integer i, let $E_i \in \mathcal{L}(\mathcal{H})$ be the projection on $\mathcal{K}_i$. If the collection $\{E_i\}_{i=1}^{\infty}$ is used to determine a unitary equivalence between $\mathcal{L}(\mathcal{K})$ and $\mathbf{A}$ as above, then the operators $C_i \in \mathcal{L}(\mathcal{H})$ which appear in the matrix X have mutually orthogonal ranges and in addition satisfy*

$$\sum \|C_i\|^2 < \infty.$$

PROOF. By definition [2, Proposition 5, p. 27], C_i is the restriction to $\mathcal{H} \oplus 0 \subset \mathcal{K}$ of an operator $F_0 T U_i \in \mathcal{L}(\mathcal{K})$, where T is as in (1) and $U_i \in \mathcal{L}(\mathcal{K})$ is a partial isometry with initial space $\mathcal{H} \oplus 0 \subset \mathcal{K}$ and final space $0 \oplus \mathcal{K}_i \subset \mathcal{K}$. Each U_i obviously has a 2×2 matrix

$$U_i = \begin{pmatrix} 0 & 0 \\ V_i & 0 \end{pmatrix},$$

where $V_i \in \mathcal{L}(\mathcal{H})$ satisfies $V_i V_i^* = E_i$ and $V_i^* V_i = 1_{\mathcal{H}}$.

By multiplying the appropriate 2×2 matrices we obtain

$$F_0 T U_i = \begin{pmatrix} CV_i & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus for $x,y \in \mathcal{H}\oplus 0$ and $i \neq j$, $(C_i x, C_j y) = (F_0 T U_i x, F_0 T U_j y) = (C V_i x, C V_j y) = 0$ since $V_i x \in \mathcal{K}_i$ and $V_j y \in \mathcal{K}_j$. This proves that C_i and C_j have orthogonal ranges. Also, for $i \geqq 2$ and $\|x\| = 1$, $\|C_i x\|^2 = (C V_i x, C V_i x) = \|C V_i x\|^2 \leqq \|C \mid \mathcal{K}_i\|^2 \leqq 1/2^i$, so that clearly

$$\sum_i \|C_i\|^2 < \infty .$$

To complete the proof of Theorem 2, it now suffices to show that if the operators $C_i \in \mathcal{L}(\mathcal{H})$ which appear in the matrix Y have mutually orthogonal ranges and satisfy

$$\sum_i \|C_i\|^2 < \infty ,$$

then Y is bounded. An easy computation which we omit shows that indeed this is the case, and in fact

$$\|Y\| \leqq \sum_i \|C_i\|^2 .$$

As an immediate corollary of Theorem 1 we obtain

COROLLARY 1.3 (HALMOS). *Any operator A on a Hilbert space $\mathcal{H}$ is the sum of two commutators.*

PROOF. Write $\mathcal{H} = \mathcal{K}\oplus\mathcal{K}$ and write A as the sum of two operators each of which vanishes on one copy of $\mathcal{K}$.

CONJECTURE. The author conjectures that Theorem 2 remains true if the restriction of compactness is removed from the operator C.

2. In [4], Halmos raised the question of whether every operator on a separable Hilbert space which is not a scalar modulo the compact operators is a commutator. If the answer to this question is yes,[1] and we denote the closed ideal of compact operators by $\mathcal{C}$, then the C^*-algebra $\mathcal{L}(\mathcal{H})/\mathcal{C}$ has the following property:

(S) Every nonscalar element of the algebra is a commutator of two elements in the algebra.

Calkin [1] imbeds $\mathcal{L}(\mathcal{H})/\mathcal{C}$ in a v.N. algebra $\mathbf{C}$ (acting on a nonseparable Hilbert space), and thus a related question is whether $\mathbf{C}$, or for that matter any v.N. algebra, has property (S).[1] A partial answer is given by the following theorem.

THEOREM 3. *Every v.N. algebra $\mathbf{A}$ which is not a factor of type* III *contains a projection $E\neq 1$ which is not a commutator in $\mathbf{A}$; thus, no such $\mathbf{A}$ has property* (S).

[1] See Remark (3) at the end of the paper.

PROOF. If $\mathbf{A}$ is not a factor, it follows from Kleinecke's result in [6] that every nonzero central projection fails to be a commutator in $\mathbf{A}$, so that it suffices to consider the case that $\mathbf{A}$ is a factor. If $\mathbf{A}$ is a finite factor, then $\mathbf{A}$ possesses a numerical-valued trace, and thus any projection with nonzero trace fails to be a commutator in $\mathbf{A}$. If $\mathbf{A}$ is a factor of type I_∞, then $\mathbf{A}$ is algebraically isomorphic to the algebra of all bounded operators on some Hilbert space $\mathfrak{K}$ and thus $\mathbf{A}$ contains the proper closed ideal $\mathfrak{I}$ of compact operators on $\mathfrak{K}$. It follows that any projection of the form $1-E$ where E is a finite-dimensional projection cannot be a commutator in $\mathbf{A}$. (If $1-E$ were a commutator in $\mathbf{A}$, then the identity element $1+\mathfrak{I}$ of the Banach algebra $\mathbf{A}/\mathfrak{I}$ would be a commutator in $\mathbf{A}/\mathfrak{I}$, which is impossible [4].) Finally, if $\mathbf{A}$ is a factor of type II_∞, let $\mathfrak{F}$ be the subset of $\mathbf{A}$ consisting of all elements which are of "finite rank" in the sense of [7, Definition 1.2.1, p. 97]. It follows from [7, Lemma 1.2.1, p. 97] that $\mathfrak{F}$ is a two-sided ideal in $\mathbf{A}$,[2] and thus the uniform closure $\mathfrak{F}_0$ of $\mathfrak{F}$ is a proper closed two-sided ideal in $\mathbf{A}$. If E is any finite projection in $\mathbf{A}$, then $E\in\mathfrak{F}$ and again $1-E$ cannot be a commutator in $\mathbf{A}$, which completes the argument.

Whether there is a type III factor with property (S) or not, the author does not know.[1] However, the following theorem indicates that perhaps every type III factor has property (S).

THEOREM 4. *If $\mathbf{A}$ is a factor of type* III, *then every $A\in\mathbf{A}$ which has a nontrivial null space is a commutator in $\mathbf{A}$. In particular, every projection $P\neq 1$ in $\mathbf{A}$ is a commutator in $\mathbf{A}$.*

PROOF. One knows that the projection E on the null space of A is an element of $\mathbf{A}$.[3] It follows from repeated application of Lemma 4.12 of [5] and the fact that all nonzero projections in $\mathbf{A}$ are equivalent that there exists a countable family $\{F_i\}$ of mutually orthogonal, equivalent projections in $\mathbf{A}$ such that

$$\sum_i F_i = E.$$

If we adjoin $1-E$ to the family $\{F_i\}$ we obtain a countable family of mutually orthogonal, equivalent projections in $\mathbf{A}$ whose sum is 1. An application of [2, Proposition 5, p. 27] yields a unitary isomorphism of $\mathbf{A}$ onto the v.N. algebra $\mathbf{B}$ of all $\aleph_0\times\aleph_0$ operator matrices with entries from the algebra $(1-E)\mathbf{A}(1-E)$, and under this isomorphism

[2] Actually in [7] only separable spaces are considered, but the transition to nonseparable spaces does not affect the validity of the lemma.

[3] Here we are assuming that every v.N. algebra contains the identity operator on the underlying Hilbert space.

A is carried onto a matrix of the form of X in Theorem 2, where $C_1 = C_2 = \cdots = 0$. But then $X = SR - RS$, just as in Theorem 2.

3. **Remarks.** (1) I wish to express my appreciation to Professor Paul Halmos for stimulating my interest in commutators and to Don Deckard for many interesting conversations on the subject.

(2) Calkin conjectured in [1] that the v.N. algebra which he constructed to contain $\mathfrak{L}(\mathfrak{H})/\mathfrak{C}$ is a factor of type III. Theorems 3 and 4 lend support to that conjecture.

(3) *Added in proof.* Since this paper was written, Arlen Brown and the author have completely settled the question of which operators on Hilbert space are commutators. Theorem 1 of the present paper proved useful in that connection. (See: *Structure theorem for commutators of operators*, Arlen Brown and Carl Pearcy, Bull. Amer. Math. Soc. **70** (1964), 779–780.) We have also proved that when $\mathfrak{H}$ is separable, every factor of type III on $\mathfrak{H}$ has property (S).

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