SOME INEQUALITIES FOR POLYNOMIALS AND THEIR ZEROS

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This note is divided into two parts. In the first part we use some results from the theory of schlicht functions to obtain inequalities involving polynomials and their zeros. Also a new proof is given to a result much used in the theory of polynomials. The second part contains some estimates for the location of the zeros of linear combinations of polynomials. A result due to Biernacki is sharpened and generalized.

I. LEMMA. *Let*

$$f(z) = \frac{z}{\prod_{k=1}^{n} (1 - \epsilon_k z)^m}.$$

Denote $\epsilon = \max_{1 \leq k \leq n} |\epsilon_k|$, *then*

(a) *If* $\epsilon \leqq 1$ *and* $m \leqq 2/n$, *the function* $f(z)$ *is regular and schlicht in the unit disc.*

(b) *If* $m = 1$, $f(z)$ *is regular and schlicht in the disc* $|z| < 1/(n-1)\epsilon$. *In general this result cannot be improved.*

PROOF. For $|z| = r$, we obtain:

$$\frac{\partial \arg f(z)}{\partial z} = \operatorname{Re}\left[z\left(\frac{1}{z} + m \sum_{k=1}^{n} \frac{\epsilon_k}{1 - \epsilon_k z}\right)\right]$$

$$(1) \qquad = 1 - \frac{nm}{2} + \frac{m}{2} \sum_{k=1}^{n} \operatorname{Re}\left(\frac{1 + \epsilon_k z}{1 - \epsilon_k z}\right)$$

$$= 1 - \frac{nm}{2} + \frac{m}{2} \sum_{k=1}^{n} \frac{1 - |\epsilon_k|^2 r^2}{|1 - \epsilon_k z|^2}.$$

Now $\partial \arg f(z)/\partial z > 0$ if $|\epsilon_k| \leqq 1$ and $1 - nm/2 \geqq 0$, for all $0 < r < 1$, which proves (a) since $\Delta \arg f(z) = 2\pi$, when z describes a circle of radius r in the politive direction. Similarly for $m = 1$,

$$(2) \qquad \partial \arg f(z)/\partial z > 0 \quad \text{if} \sum_{k=1}^{n} \frac{1 - |\epsilon_k|^2 r^2}{|1 - \epsilon_k z|^2} > n - 2.$$

Since,

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$$\min \frac{1 - |\epsilon_k|^2 r^2}{|1 - \epsilon_k z|^2} \leqq \frac{1 - \epsilon R}{1 + \epsilon R}, \quad \text{for } r \leqq R,$$

(2) is satisfied if $n(1-\epsilon R)/(1+\epsilon R) > n-2$, or $R < 1/\epsilon(n-1)$, which yields (b). The example of Corollary 1(b) provides the final part of the lemma.

THEOREM 1. *Let* $P(z) = a_n z^n + \cdots + a_1 z + a_0$ *have zeros* α_k, $k=1, \cdots, n$, $|\alpha_k| \leqq 1$, *then* $P(z) = a_n(z-\alpha)^n$, *where* $|\alpha(z)| < 1$, *for* $|z| < 1$, *and* $\alpha(z)$ *is a regular function in* $|z| > 1$.

PROOF. By a reciprocal transformation it is necessary and sufficient to prove

$$\prod_{k=1}^{n} (1 - z\alpha_k) = (1 - \alpha z)^n, \tag{3}$$

where $|\alpha(z)| < 1$ for $|z| < 1$.

For $|z| < 1$ we define $\alpha(z)$ by equation (3), choosing a single-valued branch of the nth root. It is easy to see that $\alpha(z)$ is regular in $|z| < 1$. It remains to show that $|\alpha(z)| < 1$ in $|z| < 1$.

By the lemma, $f(z) = z/(1-\alpha z)^2$ is regular and schlicht in the unit disc. Since it is also normalized by the conditions $f(0)=0$, $f'(0)=1$, it follows by a well-known estimate [2], that

$$\frac{|z|}{(1 + |z|)^2} \leqq \frac{|z|}{|1 - \alpha z|^2} \leqq \frac{|z|}{(1 - |z|)^2}. \tag{4}$$

It follows from (4) that $|\alpha(z)| < 3$ in $|z| < 1$.

Now since $1/(1-\alpha z)$ is regular in the disc $|z| < 1$, and the series

$$\frac{1}{1 - \alpha z} = 1 + \alpha z + (\alpha z)^2 + \cdots \tag{5}$$

converges for $|z| < 1/3$, it follows that the series (5) converges for all z, $|z| < 1$. Suppose that there is a point z_0, $|z_0| > 1$, such that $|\alpha(z_0)| > 1$, then by the maximum principle we may assume that $|z_0 \alpha(z_0)| > 1$ and we get a contradiction to the absolute convergence of (5) at z_0.

THEOREM 2. *Let* $\prod_{k=1}^{n} (1-\alpha_k z) = (1-\alpha z)^n$, *where* $|\alpha_k| \leqq 1$, $|\alpha(z)| \leqq 1$, *then*

$$\sum_{k=0}^{\infty} |\beta_k|^2 (2k + 1) \leqq 1$$

where the numbers β_k are the coefficients of the expansion in a Taylor series of $\alpha(z)$ in $|z|<1$.

PROOF. Since $f(z)=z/(1-\alpha z)^2$ is regular, schlicht and normalized in $|z|<1$ the same is true for the function $f_2(z)=(f(z^2))^{1/2}$. Define $F(z)=1/f_2(1/z)$, then $F(z)$ is schlicht in $|z|>1$ and has the expansion

$$F(z) = z - \frac{1}{z}\beta_0 - \frac{1}{z^3}\beta_1 - \cdots.$$

The theorem follows now by the area theorem for schlicht functions.

The following corollary is deduced easily from the theorems proved.

COROLLARY 1. *Let $P(z)=\prod_{k=1}^n(1-\alpha_k z)$, $|\alpha_k|\leqq 1/(n-1)$, then*

(a) $(1+|z|)^2\geqq|P(z)|\geqq(1-|z|)^2$ *for* $|z|\leqq 1$.

(b) $P(z)-zP'(z)\neq 0$ *for* $|z|<1$, *and this result is the best possible as the example* $P(z)=(1+z/(n-1))^n$ *shows.*

(c) *If c is such that $P(z)-z/c\neq 0$ in $|z|\leqq 1$, then*

$$(1+|z|)^2 \geqq \left|P(z)-\frac{z}{c}\right| \geqq (1-|z|)^2$$

in $|z|\leqq 1$.

(a), (b) follow by the second part of the lemma. (c) follows from the fact that the function $cf(z)(c-f(z))$ is schlicht and regular in $|z|<1$, if $f(z)=z/P(z)$.

II. LEMMA. *All the zeros of the polynomial*

$$(z+e^{i\theta})^n - 1 - nz - \cdots - \binom{n}{p-1}z^{p-1}$$

are in the disc $|z|\leqq p+1$ for $1\leqq p<n-1$; $0\leqq\theta\leqq 2\pi$.

PROOF. We use the inequality

$$(6)\qquad 1+\binom{m+q}{1}(q+1)+\cdots+\binom{m+q}{q-1}(q+1)^{q-1}<q^{m+q}.$$

For $m\geqq 3$, $q\geqq 2$, (6) was proved by Biernacki [1]. We prove (6) for $m=2$, $q\geqq 2$ and also indicate the proof for the other cases.

Let $m\geqq 2$. It is easy to verify that

$$(7)\qquad 1+\binom{m+q}{1}x+\cdots+\binom{m+q}{q-1}x^{q-1}<\binom{m+q}{q-1}(1+x)^{q-1}$$

for all $x>0$.

Substituting $x=q+1$ in (7), one deduces that (6) is true provided

$$w(m, q) = \binom{m+q}{q-1}(q+2)^{q-1}q^{-(m+q)} < 1.$$

Since $w(m, q)/w(m+1, q) > 1$ for $q > 1$, it is enough to consider the case $m=2$. In this case after obvious transformations the equivalent to (6) is the inequality $(1+2/q)^{q+2} < 1+(1+1/q)^{q+1}(5/2+4/q)$.

Since $(1+2/q)^q$ is increasing, and $(1+1/q)^{q+1}$ is decreasing, it is enough to show that $(1+2/q)^2e^2 < 1+e(5/2+4/q)$.

Set $x=1+2/p$, we get $e^2x^2-2ex<1+e/2$, hence

$$x < (1/e)[1 + (2 + e/2)^{1/2}] \quad \text{and} \quad p > 2e[1 + (2 + e/2)^{1/2} - e]^{-1}.$$

The remaining cases ($p<50$) can be verified directly.

The lemma follows now by applying Rouche's theorem, using (6). It follows easily that on the circle $|z| = p+1$,

$$|z + e^{i\theta}|^n > p^n > \left|1 + nz + \cdots + \binom{n}{p-1}z^{p-1}\right|.$$

It can be shown that the lemma is not true for $p=n-1$. We prove now

THEOREM 3. *If* $P(z)=a_nz^n+ \cdots +a_0 \neq 0$ *in* $|z|<1$, *then the polynomial* $P^*(z)=P(z)+\epsilon_na_nz^n+\epsilon_{n-1}a_{n-1}z^{n-1}+ \cdots +\epsilon_{n-p+1}z^{n-p+1} \neq 0$ *in* $|z|<1/(p+1)$, *for* $|\epsilon_k| \leqq 1$, $k=n-p+1, \cdots, n$; $1 \leqq p < n-1$.

PROOF. By a result due to Rahman [3], $P^*(z) \neq 0$ in $|z| < 1/t$, where t is the positive root of the equation

$$(t-1)^n = 1 + nt + \cdots + \binom{n}{p-1}t^{p-1}.$$

By the lemma, with $\theta=\pi$, $k=p+1$, $t \leqq p+1$, hence $1/t \geqq 1/(p+1)$. Theorem 3 generalizes a result due to Biernacki [1], obtained for $\epsilon_n=\epsilon_{n-1}= \cdots =\epsilon_{n-p+1}=-1$.

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