SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

A SIMPLE PROOF OF JACOBI'S TRIPLE PRODUCT IDENTITY

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Bellman remarks in [1, p. 42] that there are no simple proofs known of the complete triple product identity

$$\prod_{n=0}^{\infty}\left\{(1-x^{2n+2})(1+x^{2n+1}z)(1+x^{2n+1}z^{-1})\right\}=\sum_{n=-\infty}^{\infty}x^{n^2}z^n$$

with $z\neq 0$ and $|x|<1$.

However the two identities of Euler,

$$\text{(E1)}\qquad \prod_{n=0}^{\infty}(1+x^nz)=\sum_{n=0}^{\infty}\frac{x^{n(n-1)/2}z^n}{(1-x)\cdots(1-x^n)},\qquad |x|<1,$$

and

$$\text{(E2)}\qquad \prod_{n=0}^{\infty}(1+x^nz)^{-1}=\sum_{n=0}^{\infty}\frac{(-1)^nz^n}{(1-x)\cdots(1-x^n)},\quad |x|<1,\ |z|<1,$$

are rather easily established [1, p. 49]. It does not seem to have been noticed that Jacobi's triple product identity follows simply from Euler's identities.

$$\begin{aligned}\prod_{n=0}^{\infty}(1+x^{2n+1}z)&=\sum_{n=0}^{\infty}\frac{x^{n^2}z^n}{(1-x^2)\cdots(1-x^{2n})}\quad\text{(by (E1))}\\&=\sum_{n=0}^{\infty}\frac{x^{n^2}z^n\prod_{j=0}^{\infty}(1-x^{2n+2+2j})}{\prod_{j=0}^{\infty}(1-x^{2j+2})}\\&=\frac{1}{\prod_{j=0}^{\infty}(1-x^{2j+2})}\sum_{n=-\infty}^{\infty}x^{n^2}z^n\prod_{j=0}^{\infty}(1-x^{2n+2+2j})\end{aligned}$$

(all terms of the sum with negative n are zero)

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$$= \frac{1}{\prod_{j=0}^{\infty}(1 - x^{2j+2})} \sum_{n=-\infty}^{\infty} x^{n^2} z^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{m^2+m+2nm}}{(1-x^2)\cdots(1-x^{2m})} \quad \text{(by (E1))}$$

$$= \frac{1}{\prod_{j=0}^{\infty}(1 - x^{2j+2})} \sum_{m=0}^{\infty} \frac{(-1)^m (xz^{-1})^m}{(1-x^2)\cdots(1-x^{2m})} \sum_{n=-\infty}^{\infty} x^{(n+m)^2} z^{n+m}$$

$$= \frac{1}{\prod_{j=0}^{\infty}(1 - x^{2j+2})} \prod_{j=0}^{\infty}(1 + x^{2j+1}z^{-1})^{-1} \sum_{n=-\infty}^{\infty} x^{n^2} z^n$$

(using (E2) and replacing $n+m$ by n in the inner sum).

The above argument is valid provided $|x| < |z|$. The complete result for all nonzero z follows by analytic continuation.

Bibliography

1. R. Bellman, *A brief introduction to theta functions*, Holt, Rinehart and Winston, New York, 1961.

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