DECOMPOSITIONS AND REFLEXIVITY IN BANACH SPACES¹

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1. Introduction. Throughout this paper X will denote a Banach space. A sequence $\{M_i\}$ of nontrivial (not necessarily closed) subspaces of X is a *decomposition* of X provided that for each $x \in X$ there exists a unique sequence $\{x_i\}$, $x_i \in M_i$, such that $x = \lim_n \sum_1^n x_i$. Associated with each decomposition $\{M_i\}$ of X there is a sequence $\{P_i\}$ of orthogonal projections defined by $P_j(x) = x_j$ where $x = \lim_n \sum_1^n x_i$, $x_i \in M_i$. We will write: $\{M_i; P_i\}$ is a decomposition of X , $x = \sum_1^\infty x_i$, and $x = \sum_1^\infty P_i(x)$.

A decomposition is called a *Schauder decomposition* provided each of the associated projections is continuous. It can be shown [10] that this is a generalization of the Schauder basis concept. This concept was first introduced in 1949 by Grinblyum [6]. Fage, in [4] and [5], has studied this concept in Hilbert space. According to Professor Czesław Bessaga, Mazur discussed this concept at a seminar in 1953. Recently McArthur [8] has considered this concept in linear metric spaces. The author gratefully acknowledges the guidance of Dr. C. W. McArthur in the development of this material.

The purpose of this paper is to establish certain results, which are known to be valid in Banach spaces with bases, for Banach spaces with Schauder decompositions. We adopt the following notation and terminology. If $S \subset X$, $\text{sp } S$ denotes the linear manifold spanned by S , and $\text{csp } S$ denotes the closure of this set. If P is an operator on X , then $R(P)$ denotes the range of P and P^* denotes its adjoint. (X^* denotes the adjoint of X .) If $\{M_i; P_i\}$ is a decomposition of X , then $S_n = \sum_1^n P_i$ and $T_n = I - S_n$, where I is the identity operator on X . A decomposition $\{M_i; P_i\}$ is *boundedly complete* provided for each sequence $\{x_i\}$, $x_i \in M_i$, such that $\{\|\sum_1^n x_i\|\}_{n=1}^\infty$ is bounded it follows that $\{\sum_1^n x_i\}_{n=1}^\infty$ converges, and is *shrinking* provided $\lim_n \|f\|_n = 0$ for each $f \in X^*$, where

$$\|f\|_n = \sup \{ |f(x)| : x = T_n(x), \|x\| \leq 1 \},$$

i.e., $\|f\|_n$ is the norm of f restricted to $R(T_n) = \text{csp } \bigcup_{i=n+1}^\infty M_i$.

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2. Results. It is easy to see that if $\{M_i; P_i\}$ is a Schauder decomposition of X , then $\{R(P_i^*)\}$ is a Schauder decomposition of $\text{csp } \bigcup_1^\infty R(P_i^*)$. The following theorem, which is a generalization of [2, Lemma 1, p. 70], shows that in order for $\{R(P_i^*)\}$ to be a Schauder decomposition for all of X^* it is necessary and sufficient that $\{M_i; P_i\}$ be shrinking. Thus it characterizes Banach spaces with shrinking Schauder decompositions. Alaoglu [1, Theorem 2.2, p. 257] has shown that a Banach space with a boundedly complete basis is isometrically isomorphic to an adjoint space. The author does not know whether a similar result holds for Banach spaces with boundedly complete Schauder decompositions. However, Ruckle [9, Theorem IV. 4] has obtained a result very close to this.

THEOREM 1. *Let $\{M_i; P_i\}$ be a Schauder decomposition of X . The following conditions are equivalent.*

- (i) $\{R(P_i^*); P_i^*\}$ is a Schauder decomposition of X^* .
- (ii) $\lim_n \|T_n^*(f)\| = 0$ for each $f \in X^*$.
- (iii) $\{M_i; P_i\}$ is shrinking.
- (iv) $\text{csp } \bigcup_1^\infty R(P_i^*) = X^*$.

PROOF. The equivalence of (i) and (iv) follows from the result stated in the first sentence in this section. The equivalence of (i) and (ii) follows from observing that $T_n^* = \text{identity} - \sum_1^n P_i^*$, and thus $\lim_n \|T_n^*(f)\| = 0$ if and only if $\lim_n \|f - \sum_1^n P_i^*(f)\| = 0$, i.e., if and only if $\sum_1^\infty P_i^*(f) = f$ for all $f \in X^*$. The proof will be completed by showing the equivalence of (ii) and (iii). Since $T_n T_m = T_{\max(n, m)}$, for each $f \in X^*$, $\|f\|_n \leq \sup \{ \|f(T_n(x))\| : \|x\| \leq 1 \} = \|T_n^* f\|$. Thus (ii) implies (iii). Since $\{M_i; P_i\}$ is a Schauder decomposition, each M_i is closed and hence from [8, Theorem 1] it follows that there exists a constant K such that $\|S_n\| \leq K$ for all n . Hence $\|T_n\| \leq K + 1$ for all n . Thus for each n and for each $f \in X^*$, $\|T_n^*(f)\| \leq (K + 1)\|f\|_n$, and consequently (iii) implies (ii).

James [7, Theorem 1] has characterized reflexive Banach spaces with bases as those spaces whose bases are both shrinking and boundedly complete. The following theorem characterizes in an analogous manner those reflexive Banach spaces which have Schauder decompositions.

THEOREM 2. *A Banach space X with a Schauder decomposition $\{M_i; P_i\}$ is reflexive if and only if each M_i is reflexive and the decomposition $\{M_i; P_i\}$ is both shrinking and boundedly complete.*

PROOF. Assume first that X is reflexive. Since each M_i is closed, it is reflexive. To see that $\{M_i\}$ is boundedly complete, let $\{x_i\}$ be a

sequence such that $x_i \in M_i$ and $\{\|\sum_{i=1}^n x_i\|\}_{n=1}^\infty$ is bounded. For each n let $u_n = \sum_{i=1}^n x_i$. There exists a subsequence $\{u_{n(p)}\}$ of $\{u_n\}$ and a $u \in X$ such that

$$(1) \quad \lim_p f(u_{n(p)}) = f(u) \quad \text{for all } f \in X^*.$$

Hence there is a sequence $\{y_n\}$ such that

$$(2) \quad \lim_n y_n = u \quad \text{and each } y_n \text{ is a finite linear combination of the } x_i\text{'s.}$$

Thus, since P_j vanishes on $\text{csp } \bigcup_{i=1, i \neq j}^\infty M_i \equiv A$,

$$(3) \quad P_j(u) = 0 = x_j \quad \text{for each } j \text{ such that } x_j = 0.$$

If $x_j \neq 0$, then $x_j \notin A$. Hence there exists a $g \in X^*$ such that $g(x_j) \neq 0$ and $g(A) = 0$. Thus from (1), $g(u) = \lim_p g(x_j) = g(x_j)$. Hence

$$(4) \quad g(x_j) = g(u) = g\left(\sum_1^\infty P_i(u)\right) = g(P_j(u)).$$

From the second part of (2) we see that for each n there exists an a_n such that $P_j(y_n) = a_n x_j$ (a_n may be zero). Hence, from the first part of (2), $P_j(u)$ belongs to the closed subspace spanned by x_j . Thus there is a b_j such that $P_j(u) = b_j x_j$. From this and (4), $g(x_j) = b_j g(x_j)$ and hence, since $g(x_j) \neq 0$, $b_j = 1$. Consequently $P_j(u) = x_j$, and therefore, using (3), $\sum_1^\infty x_i = \sum_1^\infty P_i(u) = u$. Hence $\{M_i\}$ is boundedly complete. To see that $\{M_i\}$ is shrinking, we assume the contrary. Thus there exists $\epsilon > 0$, $g \in X^*$, and a subsequence $\{m(n)\}$ of the sequence of positive integers such that $\|g\|_{m(n)} \geq \epsilon$ for all n . For each n let x_n be such that

$$(5) \quad |g(x_n)| \geq \epsilon/2, \quad x_n = T_{m(n)}(x_n), \quad \text{and} \quad \|x_n\| \leq 1.$$

Since X is reflexive, there exists a subsequence $\{x_{n(i)}\}$ of $\{x_n\}$ which has a weak limit, say x . Thus

$$(6) \quad \lim_i f(T_{m(n(i))}(x_{n(i)})) = \lim_i f(x_{n(i)}) = f(x) = \sum_1^\infty f(P_j(x))$$

for all $f \in X^*$.

Now suppose $P_k(x) \neq 0$. Then there is an $h \in X^*$ such that $h(P_k(x)) \neq 0$ and h vanishes on $\text{csp } \bigcup_{i=1, i \neq k}^\infty M_i$. Fix t such that $m(n(t)) > k$. Then for $i > t$, $m(n(i)) > k$, and $h(T_{m(n(i))}(x_{n(i)})) = 0$. Hence from (6), $h(P_k(x)) = 0$. But this is contrary to the choice of h , and it follows that $P_i(x) = 0$ for all positive integers i . Thus $x = 0$, and therefore

$\lim_i g(x_{n(i)}) = g(0) = 0$. This contradicts (5), and it follows that $\{M_i\}$ is shrinking.

To verify the remaining part of the theorem we make use of Eberlein's result [3] that a Banach space is reflexive if and only if its unit ball is weakly sequentially compact. Let $\{y_n\}$ be a sequence in the unit ball of X . Thus $\{P_1(y_n)\}$ is a bounded sequence in the reflexive space M_1 , and hence has a subsequence with a weak limit x_1 in M_1 . The sequence of values of the operator P_2 on the points of this subsequence yields a bounded sequence in M_2 and hence has a subsequence with a weak limit x_2 in M_2 . By induction we get a sequence $\{\{y_{j,n}\}_{n=1}^\infty\}_{j=1}^\infty$ of subsequences of $\{y_n\}$ such that each sequence is a subsequence of those preceding it and such that x_i is the weak limit of $\{P_i(y_{j,n})\}_{n=1}^\infty$ for all $i \leq j$. Thus $\{y_{i,i}\}_{i=1}^\infty$ is a subsequence of $\{y_n\}$ such that for each j , x_j is the weak limit of $\{P_j(y_{i,i})\}_{i=1}^\infty$. Using this and the fact that the set of norms of the S_n 's is bounded by a number, k , one is able to show that for each $f \in X^*$, $|f(\sum_1^n x_i)| \leq k\|f\|$ for all n . Thus $\{\|\sum_1^n x_i\|\}_{n=1}^\infty$ is bounded. This, with our assumption that $\{M_i\}$ is boundedly complete and the fact that $x_i \in M_i$ for all i , guarantees the existence of a y such that

$$(7) \quad \sum_1^\infty x_i = y \quad \text{and} \quad x_i = P_i(y) \quad \text{for all } i.$$

The proof will be completed by showing that $\{y_{i,i}\}$ converges weakly to y . Let $\epsilon > 0$ be given, let $f \in X^*$, and let $K = k + 1$. If $y = 0$ let $\mu = \epsilon/4K$, and if $y \neq 0$ let $\mu = \min\{\epsilon/4K, \epsilon/4K\|y\|\}$. Since $\{M_i\}$ is shrinking, there exists an N such that

$$(8) \quad \|f\|_N < \mu.$$

For all i ,

$$(9) \quad |f(y) - f(y_{i,i})| \leq |f(S_N(y - y_{i,i}))| + |f(T_N(y))| \\ + |f(T_N(y_{i,i}))|.$$

Since $P_j(y) = x_j$ for all j and x_j is the weak limit of $\{P_j(y_{i,i})\}_{i=1}^\infty$, there exists an M such that

$$(10) \quad |f(S_N(y - y_{i,i}))| < \epsilon/2 \quad \text{for all } i \geq M.$$

For all i , $\|T_N(y_{i,i}/K)\| \leq \|T_N\| \cdot (1/K) \cdot \|y_{i,i}\| \leq (K) \cdot (1/K) \cdot 1 = 1$. Thus from (8), $|f(T_N(y_{i,i}))| = K \cdot |f(T_N(y_{i,i}/K))| < K \cdot \mu \leq \epsilon/4$. Using this and (10) in (9) we get

$$(11) \quad |f(y) - f(y_{i,i})| < 3\epsilon/4 + |f(T_N(y))| \quad \text{for all } i \geq M.$$

If $y=0$ it follows immediately from (11) that $\{y_{i,i}\}$ converges weakly to y . If $y \neq 0$, we observe that $\|T_N(y/K\|y\|)\| \leq 1$, and thus from (8) we see that $|f(T_N(y))| = K \cdot \|y\| \cdot |f(T_N(y/K\|y\|))| < K \cdot \|y\| \cdot \mu \leq \epsilon/4$. Hence, again from (11), it follows that $\{y_{i,i}\}$ converges weakly to y .

REMARK. If $\{x_i\}$ is a basis for X , then $\{M_i\}$, where M_i is the subspace of X spanned by x_i , is a Schauder decomposition of X , and the projections are given by $P_i(x) = f_i(x)x_i$, where $\{f_i\}$ is the sequence of "coefficient functionals" corresponding to the basis $\{x_i\}$. Thus as a corollary to the preceding theorem we get the result of James referred to above.

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