THE COEFFICIENTS IN AN ASYMPTOTIC EXPANSION[1]

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Put

$$e^{nz} = \sum_{r=0}^{n} \frac{(nz)^r}{r!} + \frac{(nz)^n}{n!} S_n(z),$$

where n is a positive integer and z an arbitrary complex number. Ramanujan [4, p. 26] asserted (in a different notation) that

$$S_n(1) = \frac{n!}{2}\left(\frac{e}{n}\right)^n - \frac{2}{3} + \frac{4}{135n} + O\left(\frac{1}{n^2}\right).$$

Copson [2] proved that $\{S_n(-1)\}$ is a decreasing sequence with limit $-1/2$ and derived an asymptotic series. In a recent paper, Buckholtz [1] proved that, for $k \geqq 1$,

$$S_n(z) = \sum_{r=0}^{k-1} \left(\frac{1}{n}\right)^r U_r(z) + O(n^{-k})$$

uniformly in a certain region of the z-plane. The coefficients $U_r(z)$ are determined by

$$U_r(z) = (-1)^r \left(\frac{z}{1-z} \frac{d}{dz}\right)^r \frac{z}{1-z}. \tag{1}$$

It follows from (1) that

$$U_r(z) = (-1)^r \frac{Q_r(z)}{(1-z)^{2r+1}}, \tag{2}$$

where, for $r \geqq 1$, $Q_r(z)$ is a polynomial of degree r with positive integral coefficients.

To find an explicit expression for $U_r(z)$, we put

$$F = F(z, t) = \sum_{k=0}^{\infty} U_k(z) t^k/k!. \tag{3}$$

Then, by (1),

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$$\left(\frac{z}{1-z}\frac{\partial}{\partial z}\right)F = -\sum_{k=0}^{\infty} U_{k+1}(z)t^k/k! = -\frac{\partial F}{\partial t},$$

so that

$$\frac{z}{1-z}\frac{\partial F}{\partial z} + \frac{\partial F}{\partial t} = 0. \tag{4}$$

The system

$$\frac{1-z}{z}\,dz = dt = \frac{dF}{0}$$

has the particular integrals

$$F, \qquad ze^{-z-t}.$$

Hence (4) has the solution

$$F = \phi(ze^{-z-t}), \tag{5}$$

where ϕ is arbitrary. Since

$$F(z, 0) = \frac{z}{1-z},$$

it is evident that

$$\phi(ze^{-z}) = \frac{z}{1-z}. \tag{6}$$

Now it is known [3, p. 126, no. 214] that

$$\frac{e^{\alpha z}}{1-z} = \sum_{n=0}^{\infty} \frac{(n+\alpha)^n}{n!}(ze^{-z})^n.$$

It follows that

$$\frac{z}{1-z} = \sum_{n=1}^{\infty} \frac{n^n}{n!}(ze^{-z})^n. \tag{7}$$

Comparing (7) with (6) it is clear that ϕ is determined. Thus (5) becomes

$$F(z, t) = \sum_{n=1}^{\infty} \frac{n^n}{n!}(ze^{-z-t})^n. \tag{8}$$

We have therefore

$$F(z, t) = \sum_{r=1}^{\infty} \frac{r^r}{r!} (ze^{-z})^r \sum_{k=0}^{\infty} (-1)^r \frac{r^k t^k}{k!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \sum_{r=1}^{\infty} \frac{r^{r+k}}{r!} (ze^{-z})^r,$$

so that

$$U_k(z) = (-1)^k \sum_{r=1}^{\infty} \frac{r^{r+k}}{r!} (ze^{-z})^r$$

$$= (-1)^k \sum_{r=1}^{\infty} \frac{r^{r+k}}{r!} z^r \sum_{s=0}^{\infty} (-1)^s \frac{r^s z^s}{s!}$$

$$= (-1)^k \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{r=1}^{n} (-1)^{n-r} \binom{n}{r} r^{n+k}.$$

We put

$$S(n + k, n) = \frac{1}{n!} \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} r^{n+k} \tag{9}$$

so that $S(n+k, n)$ is a Stirling number of the second kind [5, p. 33]. Thus

$$U_k(z) = (-1)^k \sum_{n=1}^{\infty} z^n S(n + k, n) \tag{10}$$

for all $k \geqq 0$.

It follows from (2) and (10) that

$$Q_k(z) = (1 - z)^{2k+1} \sum_{n=1}^{\infty} z^n S(n + k, n).$$

If we put

$$Q_k(z) = \sum_{n=1}^{k} a_{kn} z^n \qquad (k \geqq 1), \tag{11}$$

it is clear that

$$a_{kn} = \sum_{j=0}^{n} (-1)^j \binom{2k+1}{j} S(n - j + k, n - j). \tag{12}$$

For example, since

$$S(k, 1) = 1, \qquad S(k, 2) = \frac{1}{2!}(2^k - 2), \qquad S(k, 3) = \frac{1}{3!}(3^k - 3\cdot 2^k + 3),$$

we find that

$$\begin{aligned}
a_{k1} &= 1 \qquad (k \geqq 1),\\
a_{k2} &= S(2 + k, 2) - (2k + 1)S(1 + k, 1)\\
&= \tfrac{1}{2}(2^{k+2} - 2) - (2k + 1)\\
&= 2^{k+1} - 2(k + 1),\\
a_{k3} &= S(3 + k, 3) - (2k + 1)S(2 + k, 2) + \binom{2k+1}{2} S(1 + k, 1)\\
&= \tfrac{1}{6}(3^{3+k} - 3\cdot 2^{3+k} + 3) - (2k + 1)(2^{k+1} - 1) + \binom{2k+1}{2}\\
&= \tfrac{1}{2}(3^{2+k} + 1) - (2k + 3)2^{k+1} + (2k + 1)(k + 1).
\end{aligned}$$

In particular we have

$$Q_1(z) = z, \qquad Q_2(z) = z + 2z^2, \qquad Q_3(z) = z + 8z^2 + 6z^3$$

in agreement with Buckholtz.

It follows from (1) and (2) that

$$Q_{k+1}(z) = (2k + 1)zQ_k(z) - z(z - 1)Q_k'(z). \tag{13}$$

Combining (13) with (11) we get the recurrence

$$a_{kn} = na_{k-1,n} + (2k - n)a_{k-1,n-1}, \tag{14}$$

from which it is clear that the a_{kn} are positive integers for $1 \leqq n \leqq k$. By means of (14) we can easily compute the following table.

1					
1	2				
1	8	6			
1	22	58	24		
1	52	328	444	120	
1	114	1452	4400	3708	720

As a check we note that

$$Q_k(1) = \sum_{n=1}^{k} a_{kn} = 1\cdot 3\cdot 5 \cdots (2k - 1);$$

this is an immediate consequence of (13).

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ON CLASSES OF UNIVALENT CONTINUED FRACTIONS

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1. Introduction. From results of Leighton and Scott [**3**], there is a unique one-to-one correspondence between formal power series $w^{-1}+\sum_{n=2}^{\infty} c_n w^{-n}$ and C-fractions

$$(1.1)\qquad F(w) = \frac{1}{w\,-}\;\frac{a_1}{w^{\delta_1}\,-}\;\frac{a_2}{w^{\delta_2}\,-}\;\cdots\;\frac{a_n}{-\,w^{\delta_n}\,-}\;\cdots,$$

where δ_n is an integer, $\delta_1 \geqq 0$, $\delta_{n+1}+\delta_n \geqq 1$, and $a_{n+p}=0$ whenever $a_p=0$ for $n=1, 2, \cdots$. For a fixed continued fraction (1.1), let K_F denote the class of formal power series which correspond to C-fractions of the form

$$(1.2)\qquad \frac{1}{w\,-}\;\frac{a_1'}{w^{\delta_1}\,-}\;\frac{a_2'}{w^{\delta_2}\,-}\;\cdots\;\frac{a_n'}{-\,w^{\delta_n}\,-}\;\cdots,$$

where $|a_n'| \leqq |a_n|$, $n=1, 2, \cdots$. In order that each power series in K_F represent an analytic function in $|w| \geqq 1$ it is necessary and sufficient that $|a_n| \leqq g_n(1-g_{n-1})$, where $0<g_{n-1}\leqq 1$, $n=1, 2, \cdots$, and $g_{p-1}=1$ if and only if $a_p=0$ [2, p. 374]. Conditions on the parameters g_n of the chain sequence $\{g_n(1-g_{n-1})\}_{n=1}^{\infty}$ which imply that each

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