and

$$\limsup_{r\to\infty} [\mu(r)]^{1/\nu(r)} = \infty.$$

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MIXED BOUNDARY-VALUE PROBLEMS IN THE PLANE[1]

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Let R be a region in the plane bounded by a simple analytic curve C composed of N arcs $C_1 \cdots C_N$. Let a_m, b_m, f_m be analytic functions on C_m. Suppose $q(x, y)$ is non-negative in R. The mixed boundary-value problems discussed here require the determination of a solution of

$$\text{(E)} \qquad \Delta u - qu = 0 \quad \text{in } R,$$

$$\text{(A)} \qquad a_m u_n - b_m u = f_m \quad \text{on } C_m,$$

n the exterior normal. The problem is called regular if on each C_m either

$$\text{(i)} \qquad a_m > 0, \qquad b_m \geqq 0$$

or

$$\text{(ii)} \qquad a_m \equiv 0, \qquad b_m > 0.$$

This note presents an existence theorem based on integral equations. The method is an extension of the solution of the Dirichlet problem by simple layers as in [1] and [4]. It is intended also to provide information as to the behavior of u at the ends of the C_k.

THEOREM 1. *Every regular mixed problem has a unique solution.*

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Uniqueness follows from the maximum principle. To establish existence we use the Neumann function $N(P, Q)$ for (E). This is the solution having the form

$$(1) \quad N(P, Q) = A(P, Q) \log PQ + B(P, Q), \qquad A(P, P) = -1,$$

and satisfying

$$(2) \quad N_n(P, Q) = 0,\ P \in C \text{ if } q \not\equiv 0; \quad N_n(P,Q) = 2\pi/L,\ P \in C \text{ if } q \equiv 0,$$

where L is the length of C. We seek the solution in the form

$$(3) \qquad u(P) = \frac{1}{2\pi} \sum_{m=1}^{N} \int_{C_m} \sigma_m(Q) N(P, Q)\, dS.$$

Then by (1) and (2),

$$(4) \qquad u_n = \begin{cases} \sigma_k \quad \text{on } C_k & \text{if } q \not\equiv 0, \\ \sigma_k + \dfrac{1}{L} \displaystyle\sum_{m=1}^{N} \int_{C_m} \sigma_m\, dS \quad \text{on } C_k & \text{if } q \equiv 0. \end{cases}$$

Suppose first that $q \not\equiv 0$. Then conditions (A) require

$$(5) \quad a_k(P)\sigma_k(P) + \frac{b_k(P)}{2\pi} \sum_{m=1}^{N} \int_{C_m} \sigma_m(Q) N(P, Q)\, dS = f_k(P) \quad \text{on } C_k.$$

In case (i), (5) is a Fredholm system. If σ_k^0 is a solution of the homogeneous system the function u^0 formed as in (3) is a solution of the homogeneous boundary problem hence zero and by (4), $\sigma_k^0 \equiv 0$. If some $a_k \equiv 0$, the corresponding equation in (4) is of first kind. To illustrate, assume $a_2 \equiv 0$, $a_k \not\equiv 0$ for $k \neq 2$. Then the second equation in (5) is

$$(6) \qquad \begin{aligned} &\int_{C_2} \sigma_2(Q) N(P, Q)\, dS \\ &= -\left(2\pi f_2(P) + \sum_{m=1, m\neq 2}^{N} \int_{C_m} \sigma_m(Q) N(P, Q)\, dS\right) \Big/ b_2 \quad \text{on } C_2, \end{aligned}$$

the other equations remaining the same.

Let the curves C_m be parameterized by $x_m(t)$, $y_m(t)$, $0 \leqq t \leqq 1$. Since N is a solution of (E), grad $A = 0$ at $P = Q$. Thus, if $P = (x_2(t), y_2(t))$, $Q = (x_2(\tau), y_2(\tau))$, we have,

$$(7) \qquad N(P, Q) = -2 \log |t - \tau| + R_{22}(t, \tau),$$

where R_{22} has Hölder continuous derivatives. For $P \in C_2$, $Q \in C_m$, $m > 3$, $N(P, Q)$ is clearly analytic. Let $t = 0$ on both C_2 and C_3 correspond to their common end while $t = 0$ on C_1 corresponds to its common end with C_2. Then it can be seen that

$$(8)\quad N(P,Q)=\begin{cases}-2\log|t+\tau|+R_{23}(t,\tau), & P\in C_2,\ Q\in C_3,\\ -2\log|t+\tau-1|+R_{21}(t,\tau), & P\in C_2,\ Q\in C_1,\end{cases}$$

R_{23} and R_{21} again having Hölder continuous derivatives. If we incorporate the arc-length elements into σ_m, we can write (6) as

$$(9)\quad \begin{aligned}&\int_0^1 \sigma_2(\tau)\log|t-\tau|\,d\tau\\ &\quad=-\pi f_2(t)/b_2(t)+\sum_{m=1}^{N}\int_0^1 \sigma_m(\tau)K_{2m}(t,\tau)d\tau, \qquad 0<t<1.\end{aligned}$$

The left side of (9) can be inverted by a result of Carleman [**2**].

LEMMA 1. *The function*

$$\frac{1}{\pi^2\sqrt{(\tau(1-\tau))}}\left[⨍_0^1 \frac{\sqrt{(t(1-t))}h'(t)}{t-\tau}\,dt-\frac{1}{2\log 2}\int_0^1\frac{h(t)}{\sqrt{(t(1-t))}}\,dt\right]\equiv\frac{T(\tau;h)}{\sqrt{(\tau(1-\tau))}}$$

is a solution of the equation

$$\int_0^1 v(\tau)\log|t-\tau|\,d\tau=h(t),\qquad 0<t<1.$$

The symbol ⨍ indicates the principal value and the lemma is true for functions h for which the integrals exist.

The lemma can be applied to (9). As in Lemma 2 below one can show that $T(\tau;h)$ is continuous in $0\leqq\tau\leqq 1$ if h' is continuous. The only troublesome terms come from the logarithms in (8). It can be seen (again compare Lemma 2) that as $t\to 0$,

$$\int_0^1\frac{\sqrt{(s(1-s))}}{s+t}\,ds=a_0+O(\sqrt{t}),\quad ⨍_0^1\frac{\sqrt{(s(1-s))}}{s-t}\,ds=a_0+O(t).$$

Thus,

$$(10)\qquad \int_0^1\frac{\sqrt{(s(1-s))}}{(s-\tau)(t+s)}\,ds=O(\sqrt{t}/(t+\tau))\quad\text{as}\quad \tau,t\to 0.$$

We set

$$\begin{aligned}\bar\sigma_2&=\sqrt{(t(1-t))}\sigma_2, &\bar\sigma_m&=\sigma_m,\quad m\neq 2;\\ \bar f_2&=-\pi T(\tau;f_2/b_2), &\bar f_m&=f_m,\quad m\neq 2.\end{aligned}$$

Then the inverted equation (9) and equations (5) for $k\neq 2$ yield

$$(11)\qquad \bar{\sigma}_k(\tau) = \bar{f}_k(\tau) + \sum_{m=1}^{N} \int_0^1 \sigma_m(t)\overline{K}_{km}(t, \tau)\, dt, \qquad 0 < \tau < 1.$$

The kernels other than K_{21} and K_{23} behave at worst like

$$[\log |t - \tau|]/\sqrt{(t(1 - t))}$$

and from (10) one deduces,

$$K_{23} = O(\sqrt{t}/(t + \tau)) \quad \text{as } \tau, t \to 0,$$

with a similar estimate for K_{21}. It follows that the operators in (11) are completely continuous on the set of n-tuples of continuous functions. Once again the uniqueness theorem and equation (4) show that the homogeneous system has no nontrivial solution. Thus (11) has a solution and if we retrace the steps we have a solution of the boundary problem. The case $q \equiv 0$ requires only minor modifications.

Let the juncture of C_m and C_{m+1} be the origin (0, 0) with the x-axis as their common tangent and let $r^2 = x^2 + y^2$, $\theta = \text{arc}\tan(y/x)$.

THEOREM 2. *Let u be the solution of the regular mixed problem with $a_{k+1} \neq 0$. Then as $r \to 0$,*

(i) *if* $a_k \equiv 0$, $u_x \sim Cr^{-1/2}\cos(\theta/2)$, $u_y \sim Cr^{-1/2}\sin(\theta/2)$,

(ii) *if* $a_k \not\equiv 0$, $u_x \sim C' \log r$, $u_y = o(\log r)$.

This theorem is an extension of a result of Lewy [3] who proved (ii) for $q \equiv 0$. Case (i) for $q \equiv 0$ is contained in a general result of Voytuk [5]. It has been shown that σ_{k+1} is continuous. In case (ii), σ_k is also continuous but in case (i),

$$\sigma_k(t) - Ct^{-1/2} = o(t^{-1/2}) \quad \text{as } t \to 0.$$

The theorem then follows from the representation (3). The leading terms in u_x and u_y come from the two integrals

$$\int_{C_k} \sigma_k(Q) \log PQ\, dS, \qquad \int_{C_{k+1}} \sigma_{k+1}(Q) \log PQ\, dS,$$

arising from the logarithmic singularity in N. We obtain the desired estimates from the following result.

LEMMA 2. *Let $g(t) = t^\alpha + O(t^\alpha)$ as $t \to 0$, $\alpha < -1$. Then for $A > 0$,*

$$\frac{\partial}{\partial x}\int_0^A g(t)\log[(x - t) + y^2]\, dt = \begin{cases} C_\alpha r^\alpha \cos\alpha\theta + o(r^\alpha) & \text{if } \alpha < 0, \\ C_0 \log r + o(\log r) & \text{if } \alpha = 0, \end{cases}$$

$$\frac{\partial}{\partial y}\int_0^A g(t)[\log(x - t)^2 + y^2]\, dt = \begin{cases} C'_\alpha r^\alpha \sin\alpha\theta + o(r^\alpha) & \text{if } \alpha < 0, \\ o(\log r) & \text{if } \alpha = 0. \end{cases}$$

Lemma 2 can be proved by taking real and imaginary parts of the estimates for Cauchy integrals in [4] (see also [3]).

REMARKS. Extensions are possible in which the smoothness and connectivity requirements are relaxed. In particular, the results can be extended to the case in which the arcs meet at angles. The requirement that (E) satisfy a maximum principle can clearly be replaced by a uniqueness theorem. Thus the ideas can be applied for example to exterior problems for $\Delta u = -u$ where variational methods cannot be used.

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