A REMARK CONCERNING LITTLEWOOD'S TAUBERIAN THEOREM

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Let $\{a_n\}$, $n=0, 1, \cdots$, be real numbers such that the series $\sum_0^\infty a_n$ is Abel summable to s. Then, by a well-known theorem of Littlewood, if

$$(1) \qquad a_n = O\left(\frac{1}{n}\right),$$

the series converges to s. Professor Carleson has suggested to the author the question of whether the condition

$$(2) \qquad f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ is of bounded variation on } [0, 1)$$

(which plainly *implies* that $\sum a_n$ is Abel summable), implies the convergence of $\sum a_n$ under a substantially weaker Tauberian condition than (1). (Such a theorem would have useful applications in the theory of Fourier series.) We show, however, the following negative result:

THEOREM. *Given any* $\epsilon > 0$, *there is a sequence* a_n *satisfying* (2) *with*

$$a_n = O\left(\frac{1}{n^{1-\epsilon}}\right)$$

for which $\sum a_n$ *is divergent.*

PROOF. We take $a_0=0$, $a_n = n^{-\alpha} \cos n^\beta$ for $n=1, 2, \cdots$, where $\beta = 1/2k$ (k a positive integer) and $\alpha = 1-\beta$. Note first that the series $\sum a_n$ diverges because the sum of consecutive terms in a block of terms having like sign does not tend to zero, as is readily verified. Hence the theorem will be proved if for every k the function

$$(3) \qquad F(y) = \sum_{n=1}^{\infty} n^{-\alpha} \cos n^\beta e^{-ny}$$

is of bounded variation (B. V.) on $[0, \infty)$. Setting

$$(4) \qquad G(t, y) = t^{-\alpha} \cos t^\beta e^{-ty},$$

we have

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$$F(y) = \sum_{n=1}^{\infty} G(n, y) = \int_{1-0}^{\infty} G(t, y)\, d[t] = -\int_{1}^{\infty} [t] \frac{\partial G(t, y)}{\partial t} dt$$

$$= G(1, y) + \int_{1}^{\infty} G(t, y)\, dt + \int_{1}^{\infty} (t - [t]) \frac{\partial G(t, y)}{\partial t} dt$$

and we show that each of the three terms on the right is B. V.

(i) $G(1, y) = e^{-y} \in$ B. V.

(ii) Consider next the last term on the right. Call it $H(y)$. Then

$$H'(y) = \int_{1}^{\infty} (t - [t]) \frac{\partial^2 G(t, y)}{\partial t \partial y} dt,$$

$$\int_{0}^{\infty} |H'(y)|\, dy \leqq \int_{0}^{\infty} \int_{1}^{\infty} \left| \frac{\partial^2 G(t, y)}{\partial t \partial y} \right| dt dy.$$

From the estimate $|\partial^2 G(t, y)/\partial t \partial y| \leqq (2\beta t^{2\beta - 1} + y t^{\beta}) e^{-ty}$ $(t \geqq 1)$, the finiteness of the double integral follows, and so $H \in$ B. V.

(iii) We are left finally with $\int_1^\infty G(t, y)\, dt$, and it suffices to study instead $\int_0^\infty G(t, y)\, dt$, since

$$\int_{0}^{\infty} \left| \frac{\partial}{\partial y} \int_{0}^{1} G(t, y)\, dt \right| dy \leqq \int_{0}^{\infty} \int_{0}^{1} t^{1-\alpha} e^{-ty}\, dt dy < \infty.$$

Now, $\int_0^\infty G(t, y)\, dt = \int_0^\infty t^{-\alpha} \cos t^{\beta} e^{-ty}\, dt = 2kz \int_0^\infty e^{-u^{2k}} \cos zu\, du$, where we have set $u = (ty)^{\beta}$, $z = y^{-\beta}$, and $\beta^{-1} = 2k$. It is enough to verify that the last integral represents a function of B. V. for $0 \leqq z < \infty$, and for this it suffices to remark that its derivative is of class L^1, since, in fact, both $\int_0^\infty e^{-u^{2k}} \cos zu\, du$ and $\int_0^\infty u e^{-u^{2k}} \sin zu\, du$ fall off at ∞ faster than any power of z^{-1}, being Fourier transforms of functions on $(-\infty, \infty)$ possessing L^1 derivatives of every order. The theorem is proved.

REMARK. The divergence of the above series, as well as the fact that it is Abel summable, follows from [1, Theorem 84], which implies that for any $k > -1$, $\sum_1^\infty n^{-\alpha} e^{in^{\beta}}$ is summable (C, k) if and only if $(k+1)\beta + \alpha > 1$.

BIBLIOGRAPHY

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