SUBSPACES OF $C(H)$ WHICH ARE DIRECT FACTORS OF $C(H)$[1]

DAVID W. DEAN

0. **Introduction.** Let X be a subspace of $C(H)$ which contains the subspace $C_\infty(H, A)$ of functions in $C(H)$ which vanish on the subset A of H. In this note we examine when X is a direct factor of $C(H)$. In particular, if $X = C_\infty(H, A)$, it is shown that X is a direct factor of $C(H)$ if and only if there is a continuous linear mapping $E: C(A) \to C(H)$ such that Ef is an extension of f for each f in $C(A)$. If P is a continuous projection of $C(H)$ onto X it is shown, under fairly general conditions, that the complementary projection, $I-P$, must have norm $\|P\|-1$ (Theorem 2 below).

These results are used to study certain spaces $(C(K), P_\lambda)$. In particular, let X be the space of functions in $C(H)$ which are constant on A. If $C(H)$ is a P_1-space and if X is a P_λ-space it is shown that $C(A)$ is a P_λ-space. It is further shown (Theorem 3 below) that $C(A)$ then contains an isometric copy of m, the space of bounded sequences. This result is used to show that certain subspaces of m are isomorphic to m.

1. **Notation and examples.** Throughout this paper H is a compact Hausdorff space and A is a closed subspace. If the closure of each open set is open then H is said to be extremally disconnected. The Banach space of continuous functions on H with the supremum norm is called $C(H)$. The subspace of $C(H)$ of functions constant on A will be denoted by $C(H, A)$ while $C_\infty(H, A)$ stands for the subspace of functions vanishing on A. If $C(H)$ is a direct factor of every Banach space Z containing $C(H)$, that is, if one can always write $Z = C(H) \oplus Y$ for some closed subspace Y of such Z, then $C(H)$ is called a P-space. Thus if $C(H)$ is a P-space there is always a continuous projection P from $Z \supset C(H)$ onto $C(H)$. If $\|P\| \leqq \lambda$ is always possible, say that $C(H)$ is a P_λ-space and write $(C(H), P_\lambda)$. Every P-space is a P_λ-space for some λ and $C(H)$ is a P_λ-space if and only if for every $Z_1 \supset Z_2$ and $T: Z_2 \to C(H)$, T a continuous linear mapping, there is an extension $T_1: Z_1 \to C(H)$ such that $\|T_1\| \leqq \lambda \|T\|$ [3, pp. 94, 95]. Kelley [7] has shown that $(C(H), P_1)$ if and only if H is ex-

Presented to the Society, April 20, 1964; received by the editors August 29, 1963 and, in revised form, October 25, 1963.

[1] This research was supported by the United States Air Force under Grants AF AFOSR 61-51 and 62-162, monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

tremally disconnected. Thus m, the space of bounded sequences, is easily seen to be a P_1-space. Grothendieck [5, p. 169] has shown that an infinite-dimensional separable Banach space is not a P-space while Phillips [9, p. 539] proved earlier there was no continuous projection from m onto c, the space of convergent sequences, and hence no projection of m onto c_0, the space of sequences converging to zero.

Two Banach spaces are said to be isomorphic if there is a one-to-one continuous linear mapping from one onto the other (with a continuous inverse) and isometric if this mapping preserves norm.

EXAMPLE 1. Let S be an extremally disconnected Hausdorff space and let $K=S\cup\{\infty\}$ be its one-point compactification, which we assume is Hausdorff. In [1] Amir has studied such $C(K)$ when $C(K)$ is a P_λ-space; and $C(K)$ is completely characterized if $\lambda<3$. Let H be the Stone-Čech compactification of S. Then H is extremally disconnected and the space $C(H, A)$ is isometric to $C(K)$ and is then P_λ if $C(K)$ is. For any H, the space $C(H, A)$ is isometric to $C(K)$, where K is the one-point compactification of $H-A$.

EXAMPLE 2. Let (S, Σ, μ) be a nonatomic finite measure space and let M be the space of bounded measurable functions on S with the supremum norm. Let M_∞ be the essentially bounded measurable functions identified, as usual, off null measure sets, with the essential supremum norm. Then M is isometric to a $C(H)$ space. The measure μ corresponds to a Radon measure ν on H and $\int_S f(s)\,d\mu(s) =\int_H g(h)\,d\nu(h)$ if f in M corresponds to g in $C(H)$ under the isometry above. Let A be the support of ν (see [2, p. 70]). Then A is closed in H (and is a category-one set). Two functions in $C(H)$ correspond to functions in the same equivalence class in M if and only if they agree on A. The space M_∞ is isometric to $C(A)$ and A is extremally disconnected.

A. Ionescu Tulcea and C. Ionescu Tulcea have shown there is a norm-one linear mapping $T: M_\infty\to M$ such that Tf is in the equivalence class $\{f\}$ for every $\{f\}\in M_\infty$ [10]. Translating to $C(H)$ there is a norm-one linear mapping E from $C(A)$ to $C(H)$ which extends each f in $C(A)$ to a g in $C(H)$; that is, $g(h)=f(h)$ if h is in A. Let $R: C(H)\to C(A)$ be the restriction mapping $Rf(a)=f(a)$ for every a in A. For each g in $C(H)$ let $Pg=g-ERg$. Then P is a norm-two projection of $C(H)$ onto $C_\infty(H, A)$, the subspace of $C(H)$ isometric to the space of μ-null functions.

2. **The equation** $C(H)=C_\infty(H, A)\oplus Y$. In general, for any H, A, if there is a linear continuous mapping $E: C(A)\to C(H)$ which extends each f in $C(A)$ to Ef in $C(H)$, then there is a projection P of

$C(H)$ onto $C_\infty(H, A)$, $P=I-ER$, where R is the restriction mapping defined above.

THEOREM 1. *$C(H)=C_\infty(H, A)\oplus Y$ if and only if the extension mapping E exists from $C(A)$ to $C(H)$. The space Y is isomorphic to $C(A)$.*

PROOF. That E implies $C(H)=C_\infty(H, A)\oplus Y$ is shown above. Now let P be a continuous projection of $C(H)$ onto $C_\infty(H, A)$ and let Y be the subspace of f in $C(H)$ for which $Pf=0$. Clearly R is linear and $\|R\|=1$. We shall show that R is an isomorphism of Y with $C(A)$. Then R^{-1} is continuous and let $E=R^{-1}$. If g is in $C(A)$ and if f is an extension of g, let $f=f_\infty+y$, where f_∞ is in $C_\infty(H, A)$ and y is in Y. Then $Ry=g$ so R: $Y\to C(A)$ is onto. If $Ry=0$, then y vanishes on A and so y is in $C_\infty(H, A)$ also. Thus y is the 0 element of $C(H)$ and so R is one-to-one. The open mapping theorem ends the proof; however, the following computation is sufficient and useful. Assume $\|y\|=1$, and $\|Ry\|<1/\|P\|$, let x agree with Ry on A, have norm $1/\|P\|$ and have value $-1/\|P\|$ where $y=1$. Then $\|y-x\|=1+1/\|P\|$ and $y-x$ is in $C_\infty(H, A)$. But $\|P\|\,\|x\|=1\geqq\|Px\|=\|P(y-x)\|=\|y-x\|=1+1/\|P\|$, since $Py=0$ and $y-x$ is in $C_\infty(H, A)$. This contradiction shows $\|Ry\|\geqq(1/\|P\|)\,\|y\|$ for each y. Q.E.D.

From the proof of Theorem 1 one easily proves the following.

COROLLARY 1. *Let X be a subspace of $C(H)$ which contains $C_\infty(H, A)$ and suppose one can write $C(H)=X\oplus Y$ where X and Y are closed. Then R restricted to Y is an isomorphism of Y into $C(A)$. If $Y_1=R(Y)$, then Y_1 is a direct factor of $C(A)$.*

THEOREM 2. *With the hypothesis of Corollary 1 let P be the indicated projection of $C(H)$ onto X. Suppose also that H is totally disconnected and that A is nowhere dense. Then the complementary projection of $C(H)$ onto Y has norm $\|P\|-1$. That is, $\|I-P\|=\|P\|-1$.*

PROOF. Let $\|I-P\|=\lambda\geqq 1$. Let $\epsilon>0$ and choose f in $C(H)$ such that $\|f\|=1$ and such that $\|(I-P)f\|>\lambda-\epsilon$. Because A is nowhere dense, there is a point h in $H-A$ such that $|(I-P)f(h)|>\lambda-2\epsilon$. Without loss of generality, assume $(I-P)f(h)>\lambda-2\epsilon$.

Since H is totally disconnected, we may find an open and closed neighborhood U of h which does not meet A. Define f_1 by $f_1=1+f$ on U and $f_1=0$ off U. Let $g=f_1-Pf$. Since f_1 is in $C_\infty(H, A)$, we have that $Pf_1=f_1$, and so $Pg=g$. Consider the function $g-(I-P)f$ which is 1 on U, $-f$ off U, and has norm 1. Then $P(g-(I-P)f)=Pg=g$. Since $\|g\|\geqq|g(h)|=|1+f(h)-Pf(h)|>1+\lambda-2\epsilon$, we have $\|P\|\geqq 1+\lambda-2\epsilon$. Q.E.D.

Corollary 2. *With the hypotheses of Theorem* 2, *if* $\|P\|<2$ *then* $X=C(H)$.

Proof. $\|I-P\|<1$ so that $I-P$ is the 0 projection, or $I=P$.

3. **The equation** $C(H)=C(H, A)\oplus Y$. Throughout this section, S is a locally compact, extremally disconnected Hausdorff space and H is the Stone-Čech compactification of S. Let $A=H-S$.

If K is a Hausdorff compactification of S, then $C(K)$ is isometric to a sublattice of $C(H)$ containing $C_\infty(H, A)$. In particular, if K is the one-point compactification of S, then $C(K)$ is isometric to $C(H, A)$. The isometry may be constructed as follows. Let ρ be a continuous function from H onto K. Let T: $C(K)\rightarrow C(H)$ be defined by setting, for each f in $C(K)$, $Tf(h)=f(\rho(h))$ for every h in H.

Since $(C(H), P_1)$, one concludes from Corollary 2 that the conditions $(C(K), P_\lambda)$ and $\lambda<2$ together imply that $C(K)$ is a P_1-space. This result is proved by Amir in [1], where it is also shown that if $(C(K), P_\lambda)$, then K is a compactification of such an S.

Corollary 3. *If K is an extremally disconnected compactification of S, then K is homeomorphic to H.*

Proof. Let T be the isometry defined above and let $X=T(C(H))$. If P is a projection of $C(H)$ onto X with $\|P\|=1$, then $\|I-P\|=0$ so that $I=P$. Thus $X=C(H)$ from which it easily follows that ρ is a homeomorphism.

Corollary 4. *If K is the one-point compactification of S and if $(C(K), P_\lambda)$, then $(C(A), P_\lambda)$.*

Proof. Since $C(K)$ is isometric to $C(H, A)$, $C(H, A)$ is a P_λ-space. Then $C_\infty(H, A)$ is a $P_{\lambda+1}$-space, and write $C(H)=C_\infty(H, A)\oplus Y$. Let R be the restriction mapping of Y onto $C(A)$ and E the inverse mapping (as in the proof of Theorem 1). If Z is a Banach space containing $C(A)$, the mapping E: $C(A)\rightarrow C(H)$ has an extension T: $Z\rightarrow C(H)$ such that $\|T\|=\|E\|$. Then RT is a projection of Z onto $C(A)$ and $\|RT\|\leqq\|R\|\,\|T\|=\|E\|$. Now $ER=I-P$, where P is the indicated projection of $C(H)$ onto $C_\infty(H, A)$. Hence $\|ER\|=\|I-P\|\leqq\lambda+1-1=\lambda$ if $\|P\|\leqq\lambda+1$, by Theorem 2. It remains to show $\|E\|=\|ER\|$. Let f be in $C(A)$ and $\|f\|=1$. Let $Rg=f$ and $\|g\|=1$. Then $ERg=Ef$ and we conclude that $\|ER\|$ is as large as $\|E\|$.

Corollary 5. *If $(C_\infty(H, A), P_\lambda)$ and if $\lambda<3$, then $C(A)$ is a P_1-space.*

Proof. Using the proof of Corollary 4, $\|E\|=\|ER\|=\lambda-1<2$, and so $(C(A), P_{\lambda-1})$. The remarks preceding Corollary 3 conclude the proof.

THEOREM 3. *$C(A)$ contains an isometric image of m, the space of bounded sequences, if A is infinite.*

To prove Theorem 3 we require two lemmas. The first is Theorem 1 in [1] for which we give a different proof.

LEMMA 1 (AMIR). *If B is a compact Hausdorff space and if $(C(B), P_\lambda)$, then no infinite sequence of distinct points in B converges.*

PROOF. Let b_n be such a sequence and suppose $b_n \to b$. Without loss of generality, assume $b_n = b$ for no n. Then we may find a sequence of mutually disjoint open sets $\{U_n\}$ such that b_n is in U_n for each n. Let f_n vanish off U_n and $\|f_n\| = 1 = f_n(b_n)$, for each n. Then c_0 is embedded isometrically in $C(B)$ by letting the sequence $\{t_n\}$ in c_0 correspond to the function $\sum_1^\infty t_n f_n$ in $C(B)$. For each f in $C(B)$, let $Pf = \sum_1^\infty (f(b_n) - f(b))f_n$. Then P is a continuous projection of $C(B)$ onto a subspace isometric to c_0, contradicting Phillips' result that c_0 is not a P_λ-space [9].

COROLLARY 6. *If B contains an infinite sequence of distinct points which converges, then $C(B)$ has a direct factor which is isometric to c_0.*

LEMMA 2. *H contains a mutually disjoint sequence of open and closed sets $\{V_n\}$ such that for each n, $V_n \cap A$ is nonempty.*

PROOF. Fix a_1 in A and let U_1 be a neighborhood of a_1 such that $A - U_1$ is infinite (if no such U_1 exists, every sequence in A converges to a_1, contradicting Lemma 1). Let f_1 in $C(A)$ vanish on $A - U_1$ and $\|f_1\| = 1 = f_1(a_1)$. Let g_1 be an extension to $C(H)$ of f_1 and $\|g_1\| = 1$. The closure V_1 of the set $\{h \mid g_1(h) > \frac{1}{2}\}$ is open and closed, and intersects A in a set W_1 which contains a_1 and is open and closed in A. Moreover, $A - W_1$ is open, closed and infinite. Suppose mutually disjoint open and closed sets $V_1, \cdots, V_{n-1}$ are chosen, each meeting A, in nonempty $W_1, \cdots, W_{n-1}$, and that $A - \bigcup_1^{n-1} W_j$ is infinite. Choose a_n in $A - \bigcup_1^{n-1} W_j$ and a neighborhood U_n of a_n contained in $A - \bigcup_1^{n-1} W_j$ such that $A - \bigcup_1^{n-1} W_j - U_n$ is infinite. (If no such U_n exists, every sequence in $A - \bigcup_1^{n-1} W_j^1$ converges to a_n.) Let f_n in $C(A)$ vanish off U_n and $\|f_n\| = 1 = f_n(a_n)$. Let g_n be an extension of f_n to $C(H)$. The closure of $\{h \mid g_n(h) > \frac{1}{2}\}$ is open and closed in H. Its intersection with $H - \bigcup_1^{n-1} W_j$ is an open and closed set V_n which meets A in nonempty W_n. Moreover, $A - \bigcup_1^n W_j$ is infinite. By induction the desired sequence exists.

From the proof we have

COROLLARY 7. *If B is an infinite closed subspace of H then $C(B)$ contains an isometric copy of m.*

Proof. No subsequence in B converges since H is extremally disconnected. Then use the proof of Lemma 2.

Proof of Theorem 3. Following James [6, p. 900] (see also [4, p. 390]) m is embedded isometrically in $C(H)$ as the subspace $\bar{m}$ of functions constant on each V_i and if f corresponds to the sequence t_i, then $f(h)=t_i$ if h is in V_i. R restricted to $\bar{m}$ embeds $\bar{m}$ isometrically in $C(A)$.

Corollary 8. *If $H=\beta(N)$, the Stone-Čech compactification of the positive integers N, if $A\subset\beta(N)-N$, and if $(C(H, A), P_\lambda)$, then $C(A)$ is isomorphic to m.*

Proof. $C(H)$ is isometric to m and $E: C(A)\rightarrow C(H)$ is an isomorphism into. By Theorem 3, m is isometric to a subspace of $C(A)$. Thus $\dim_l(C(A))=\dim_l(m)$. By the corollary to Theorem 6 in [4] (or see [7]), since $(C(A), P_\lambda)$ for some λ, m and $C(A)$ are isomorphic.

Theorem 4. *With the hypotheses of Corollary 8, $C(H, A)$ and $C_\infty(H, A)$ are isomorphic to m.*

Proof. Since both are P-spaces and subspaces of $C(H)$, it is enough to show m is isometric to a subspace of $C_\infty(H, A)$. Now $A\neq\beta(N)-N$ since $C_\infty(H, \beta(N)-N)=c_0$. Let h be in $\beta(N)-N-A$. Let V be an open and closed neighborhood of h not meeting A. Then V is infinite, containing an infinite number of integers; and $C_\infty(H, H-V)\subset C_\infty(H, A)$. Since $(C_\infty(H, H-V), P_1)$, it contains an isometric copy of m [6].

Bibliography

1. D. Amir, *Continuous functions' spaces with the bounded extension property*, Bull. Res. Council Israel Sect. F **10F** (1962), 133–138.

2. N. Bourbaki, *Intégration*, Hermann, Paris, 1952; Chapitres I–IV.

3. M. M. Day, *Normed linear spaces*, Academic Press, New York, 1962.

4. D. W. Dean, *Projections in certain continuous function spaces*, Canad. J. Math. **14** (1962), 385–401.

5. A. Grothendieck, *Sur les applications linéaires faiblement compactes d'espaces du type* $C(K)$, Canad. J. Math. **5** (1953), 129–173.

6. R. C. James, *Projections in the space* (m), Proc. Amer. Math. Soc. **6** (1955), 899–902.

7. J. L. Kelley, *Banach spaces with the extension property*, Trans. Amer. Math. Soc. **72** (1952), 323–326.

8. A. Pelczynski, *Projections in certain Banach spaces*, Studia Math. **19** (1960), 209–228.

9. R. S. Phillips, *On linear transformations*, Trans. Amer. Math. Soc. **48** (1940), 516–541.

10. A. Ionescu Tulcea and C. Ionescu Tulcea, *On the lifting property*. I, J. Math. Anal. Appl. **3** (1961), 537–546.

Duke University