ON A PROBLEM OF GRÜNBAUM

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In [1, p. 249] Branko Grünbaum asked whether the set of all points which bisect three or more chords of a planar convex body was connected or even convex. In this note we show that this set is always polygonally connected but not necessarily convex.

Let C be a given planar convex body. For each $\alpha \in [0, \pi)$, let L_α , called a curve, denote the set of all points which are midpoints of chords of direction α which intersect the interior of C . Then each L_α is a subset of C^0 (the interior of C) and is the graph of a real-valued function which is the difference of two convex functions on some open interval. Let $\mathcal{L}$ denote the collection of all L_α for $\alpha \in [0, \pi)$.

First we establish the following lemma whose proof is clear and straightforward and hence, will be omitted.

LEMMA 1. (a) *Each two curves in $\mathcal{L}$ intersect in either a closed line segment or in a single point.*

(b) *Two curves in $\mathcal{L}$ intersect in a closed line segment if and only if there exists a pair of parallel line segments in the boundary of C .*

(c) *If there is a point x common to k curves in $\mathcal{L}$, then x bisects at least k chords of C .*

(d) *If $L_\alpha \cap L_\beta$ is not a single point, then each point of $L_\alpha \cap L_\beta$ bisects infinitely many chords.*

If L_α , L_β , and L_γ are three noncurrent curves in $\mathcal{L}$ (i.e., $L_\alpha \cap L_\beta \cap L_\gamma = \emptyset$), then by the triangle determined by L_α , L_β and L_γ we mean the nonempty closed set which is the union of the sets $L_\alpha \cap L_\beta$, $L_\beta \cap L_\gamma$ and $L_\alpha \cap L_\gamma$ with the closure of that component of $C - L_\alpha - L_\beta - L_\gamma$ whose closure lies entirely in C^0 .

LEMMA 2. *Each point interior to the triangle determined by three non-concurrent curves in $\mathcal{L}$ bisects at least three chords of C .*

PROOF. Let L_α and L_β be in $\mathcal{L}$. And let x be interior to one of the four components of $C^0 - L_\alpha - L_\beta$. Then the directed distances from x to $M_\alpha \cap L_\alpha$ and $M_\beta \cap L_\beta$ will be of different sign, where M_α and M_β are the lines through x in direction α and β , respectively. Since the directed distance is continuous, there will be a γ "between" α and β such that $x \in M_\gamma \cap L_\gamma$. Applying this argument to each of the three pairs of lines selected from those determining the given triangle, we easily obtain three distinct chords which x bisects.

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LEMMA 3. *Let T be any triangle formed by three nonconcurrent curves in $\mathcal{L}$. Then any point x common to two of the curves, say L_α and L_β , can be joined to any point in T^0 by a polygonal arc lying in $T^0 \cup (L_\alpha \cap L_\beta)$.*

PROOF. First of all, T^0 being a connected open subset of the plane is polygonally connected. Next let z be the point in $L_\alpha \cap L_\beta \cap \bar{T}^0$. Then z will be accessible to T^0 by a line segment (i.e., there exists a $w \in T^0$ such that $(z, w] \subseteq T^0$). For in case $L_\alpha \cap L_\beta$ is not a single point, the extension of the line segment $L_\alpha \cap L_\beta$ into T^0 is easily seen to yield such a line segment. And in case $L_\alpha \cap L_\beta = \{z\}$, there clearly exist distinct lines M and N through z which are tangent to C_α and C_β , respectively (any curve in $\mathcal{L}$ being the graph of a function of bounded variation has left and right tangents everywhere). Then it easily follows that the extension into T^0 of any line "between" M and N and passing through z yields such a line segment.

Now let $x \in L_\alpha \cap L_\beta$ and $y \in T^0$ and let z and w be as specified above. Then $[x, z] \cup [z, w] \cup P(w, y)$, where $P(w, y)$ is a polygonal arc joining w to y in T^0 , will have a polygonal arc as a subset which joins x to y in $(L_\alpha \cap L_\beta) \cup T^0$.

THEOREM. *The set of points which bisect three or more chords of a planar convex body is polygonally connected.*

PROOF. Let x and y be two such points. Then clearly there will exist two triangles, S and T , whose interiors intersect such that x (resp. y) belongs to two "sides" of S (resp. T). Letting $w \in S^0 \cap T^0$ we can, by applying Lemma 3, obtain the appropriate polygonal arcs joining w to x and y , respectively. From these two arcs we can obtain a polygonal arc joining x and y which contains, by Lemmas 1 and 2, only points which bisect three or more chords.

As an example to show that the set of points, B , bisecting three or more chords may fail to be convex, take a regular pentagon P . It can be shown without difficulty that B has the shape of a "five-pointed star." Specifically, let $v_1, \dots, v_5$ denote (in order) the vertices of P ; let M_i be the chord joining v_i to the midpoint of the opposite side; let y_i be the point of intersection of M_i and the chord $[v_{i-1}, v_{i+1}] \pmod{5}$; and let w_i be the point of intersection of M_i and the segment $[y_{i-2}, y_{i-3}] \pmod{5}$. Then it can be shown that B is contained in the closed convex hull of $y_1, \dots, y_5$. Moreover, $y_i \in \bar{B}$ and $w_i \notin \bar{B}$ for each i . Hence, B is not convex.

BIBLIOGRAPHY

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