ON THE INDECOMPOSABILITY OF TORSION-FREE ABELIAN GROUPS

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1. We consider the following question posed by E. Weinberg [4, §5.2]: Does there exist a torsion-free abelian group of cardinality greater than the continuum ($\aleph$) with the property that each pure subgroup is (directly) indecomposable? In §2 we answer this question negatively for a large class of groups which contains, most notably, the class of homogeneous groups. In §3 we characterize, in terms of indecomposability, the pure and p -pure subgroups of the p -adic integers.

Throughout this note all groups are abelian. The notation follows the usage in [2].

2. DEFINITION. A *purely indecomposable* torsion-free group is a group every pure subgroup of which is indecomposable.

We shall prove the

THEOREM. *Let G be a torsion-free group of cardinality m with the property: In the type set of G there exists a type τ maximal with respect to the property that m elements of G have this type. If G is purely indecomposable, then $m \leq \aleph$.*

The proof of this theorem is accomplished in four steps, the first three of which are lemmas.

LEMMA 1. *Let G be a purely indecomposable torsion-free group. Let $g \in G$ and put $\bar{G} = G/\{g\}_*$. Then, for each $g_\alpha \in G$ with $T(g_\alpha) = T(g)$, $T(g_\alpha) < T(\bar{g}_\alpha)$.*

PROOF. We use a result of Baer [2, Lemma 46.3] which states that if H is torsion free and if S is a pure subgroup of H such that (i) H/S has rank 1, (ii) the type of H/S is τ , and (iii) each element of H not in S has type τ , then S is a direct summand of H . To prove the lemma, let $S = \{g\}_*$ and $H = \{g_\alpha, g\}_*$. Since G is purely indecomposable and H is pure in G , there must exist an element $x \in H$, $x \notin S$, such that $T(x) > T(g_\alpha)$. Then for some integers k, m and n , $kx = mg_\alpha + ng$, and $T(\bar{g}_\alpha) = T(\bar{x}) \geq T(x) > T(g_\alpha)$.¹

LEMMA 2. *Let G be torsion free, $g \in G$, and $\bar{G} = G/\{g\}_*$. Suppose there is an independent subset S consisting of $m > \aleph$ elements of G all*

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¹ The author is indebted to the referee for this simple proof which replaces the lengthy original one.

having the same height and an infinite set π of primes such that $h_p(g_\alpha) < h_p(\bar{g}_\alpha)$ for each $g_\alpha \in S$ and $p \in \pi$. Then S contains a subset T of cardinality m such that whenever $g_\alpha, g_\beta \in T$, $g_\alpha \neq g_\beta$, then $T(g_\alpha - g_\beta) > T(g_\alpha)$.

PROOF. Fix $g_\alpha \in S$ and $p \in \pi$. Let $h_p(g_\alpha) = k \geq 0$. Then $h_p(\bar{g}_\alpha) \geq k+1$ and there is $r_{\alpha,p} \in Q$ with $p^{k+1}x = g_\alpha + r_{\alpha,p}g$, $x \in G$. Thus, with each g_α we associate a sequence $\langle r_{\alpha,p} \rangle_{p \in \pi}$ of rational numbers. Considering cardinalities, we conclude that there are m elements of S , all of which are associated with the same sequence. Denote the set of these elements by T . Then, whenever $g_\alpha, g_\beta \in T$, $g_\alpha \neq g_\beta$,

$$h_p(g_\alpha - g_\beta) \geq \min\{h_p(g_\alpha + r_{\alpha,p}g), h_p(g_\beta + r_{\beta,p}g)\} > h_p(g_\alpha),$$

for each $p \in \pi$. Hence $T(g_\alpha - g_\beta) > T(g_\alpha)$.

LEMMA 3. Let G be torsion-free, $g \in G$, and $\bar{G} = G/\{g\}_*$. Suppose there is an independent subset S of G of cardinality $m > \aleph$ and a prime p such that $h_p(\bar{g}_\alpha) = \infty$ for every $g_\alpha \in S$. Then there is a subset T of S of cardinality m such that whenever $g_\alpha, g_\beta \in T$, $g_\alpha \neq g_\beta$, then $h_p(g_\alpha - g_\beta) = \infty$.

PROOF. Fix $g_\alpha \in S$. For each positive integer n there is $r_{\alpha,n} \in Q$ with $h_p(g_\alpha + r_{\alpha,n}g) \geq n$. As in Lemma 2, associate with each $g_\alpha \in S$ the sequence $\langle r_{\alpha,n} \rangle$. Considering cardinalities, we conclude there are m elements of S which are associated with the same sequence. Denote this set of elements by T . Then, whenever $g_\alpha, g_\beta \in T$, $g_\alpha \neq g_\beta$,

$$h_p(g_\alpha - g_\beta) \geq \min\{h_p(g_\alpha + r_{\alpha,n}g), h_p(g_\beta + r_{\beta,n}g)\} \geq n,$$

for all positive integers n . Hence $h_p(g_\alpha - g_\beta) = \infty$.

PROOF OF THE THEOREM. Suppose $m > \aleph$. Then there is a set S of m independent elements, all having the same height H and type τ maximal with respect to the property stated in the theorem. Let g be one of these. By Lemma 1 the type of each $g_\alpha \in S$ different from g increases in passing to $G/\{g\}_*$. It follows that there exists a subset T of S as described in Lemma 2 or Lemma 3. In either case the existence of such a set T contradicts the maximality of the type τ .

3. We conclude with the following characterizations of pure and p -pure subgroups of the additive group P of p -adic integers.

PROPOSITION 1. A torsion-free group G is isomorphic to a pure subgroup of P if and only if G is homogeneous with the type of P and is purely indecomposable.

PROPOSITION 2. A torsion-free group G is isomorphic to a p -pure subgroup of P if and only if G is p -reduced and p -purely indecomposable; i.e., G has no nonzero subgroup H such that $pH = H$ and every p -pure subgroup of G is indecomposable.

It is convenient to utilize the notion of a p -basic subgroup of a torsion-free group as presented in [3]. We refer the reader to this paper for definitions and properties of p -basic subgroups.

LEMMA. *A torsion-free group G is p -reduced and has a cyclic p -basic subgroup if and only if G is isomorphic to a p -pure subgroup of P .*

PROOF. Let $\{x\}$ be a p -basic subgroup of G . Define $\phi': \{x\} \rightarrow P$ by $\phi'(x) = \pi \in P$, $\pi \neq 0$. Then we obtain the exact sequence $0 = \text{Hom}(G/\{x\}, P) \rightarrow \text{Hom}(G, P) \rightarrow \text{Hom}(\{x\}, P) \rightarrow \text{Ext}(G/\{x\}, P) = 0$. The left-hand equality follows since $G/\{x\}$ is p -divisible, and the right-hand equality since $\{x\}$ is p -pure in G (see [1, §3]). Consequently, ϕ' has a unique extension to ϕ mapping G into P . Since $\text{Ker } \phi$ is pure in G , $\rho_p(\text{Ker } \phi) + \rho_p(\text{Im } \phi) = \rho_p(G) = 1$, where ρ_p denotes the cardinality of a maximal p -independent subset [3, Lemma 4]. Since $\text{Im } \phi \neq 0$, $\rho_p(\text{Ker } \phi) = 0$ and, therefore, ϕ is an isomorphism of G into P . Moreover, $\text{Hom}(G, P) \cong P$ which implies that G is isomorphic to a p -pure subgroup of P [1]. Conversely, if G is a p -pure subgroup of P , then G contains a p -adic unit u and $G/\{u\}$ is p -pure in

$$P/\{u\} \cong \sum Q + \sum_{q \neq p} C(q^\infty).$$

Hence $G/\{u\}$ is p -divisible and since $\{u\}$ is p -pure in G , $\{u\}$ is a p -basic subgroup of G .

PROOF OF PROPOSITION 1. The necessity is well known [2, Theorem 43.1]. Conversely, if $g \in G$ is an element of zero p -height, then since, by Lemma 1, each element of $G/\{g\}_*$ has type greater than the type of P , it follows that $G/\{g\}_*$ is divisible. Hence $\{g\}$ is p -basic in G and G is isomorphic to a p -pure subgroup of P , which is pure since G has the type of P .

PROOF OF PROPOSITION 2. If G is p -reduced and p -purely indecomposable, then G has a cyclic p -basic subgroup and, by the lemma, G is isomorphic to a p -pure subgroup of P . The converse was proven in [1, §3].

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