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A REPRESENTATION THEOREM FOR CONTINUOUS FUNCTIONS OF SEVERAL VARIABLES

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Let E^n denote the n-dimensional unit cube in Euclidean space; designate the closed unit interval, $[0, 1]$, by E. We prove in this note the following

THEOREM. *For any natural number n, $n \geqq 2$, there exist real, monotonic increasing functions, $h^p(x)$, $1 \leqq p \leqq n$, dependent on n, and having the following properties:*

(i) *The function*

$$\sum_{1 \leq p \leq n} h^p(x_p)$$

separates all points of E^n:

$$\sum_{1 \leq p \leq n} h^p(x_p) \neq \sum_{1 \leq p \leq n} h^p(y_p)$$

unless $x_p = y_p$ for all admitted values of p.

(ii) *Every continuous function of n variables, $f(x_1, \cdots, x_n)$, with domain E^n, can be represented in the form*

$$f(x_1, \cdots, x_n) = g\left[\sum_{1 \leq p \leq n} h^p(x_p)\right]. \tag{1}$$

Clearly, the function g will, in general, be discontinuous.

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As shown by V. I. Arnol'd [1], even a simple function such as $f(x, y) = xy$ cannot be represented in the form (1) if the functions g and h^p are required to be continuous.

PROOF. We represent the real numbers in the interval E to the base n:

$$x = \sum_{1 \leq \nu \leq \infty} s_\nu \cdot n^{-\nu}, \tag{2}$$

where s_ν is an index with domain $0 \leqq s_\nu \leqq n-1$ for each ν. To have a one-to-one correspondence between the real numbers, $x \in E$, and the infinite series (2), we normalize (2) by requiring that for no $N > 1$ is $s_\nu = n-1$ for all $\nu \geqq N$, except when $s_\nu = n-1$ for all ν.

To prove (i), we set for each admitted p

$$x_p = \sum_{1 \leq \nu \leq \infty} s_{p\nu} \cdot n^{-\nu}, \tag{3}$$

where $0 \leqq s_{p\nu} \leqq n-1$, the $s_{p\nu}$ being subject to the normalizing restriction just described, and define the functions

$$h^p(x_p) = \sum_{1 \leq \nu \leq \infty} s_{p\nu} \cdot n^{-n\nu-p+1}. \tag{4}$$

Clearly, (4) is a representation of real numbers in E to the base n^n. By their construction, the functions $h^p(x)$ are monotonic increasing and bounded for $x \in E$ (and hence, continuous almost everywhere).

Let p be fixed; consider an infinite sequence $\{s_{p\nu}\}$: this sequence determines simultaneously unique numbers x_p and $h^p(x_p)$. It follows that the correspondence

$$x_p \leftrightarrow h^p(x_p) \tag{5}$$

is one-to-one for each admitted p.

Let us write

$$\sum_{1 \leq p \leq n} h^p(x_p) = n^{1-n} \sum_{1 \leq \nu \leq \infty} t_\nu \cdot n^{-n\nu}, \tag{6}$$

where

$$t_\nu = \sum_{1 \leq p \leq n} s_{p\nu} \cdot n^{n-p}. \tag{7}$$

We first show that the normalizing restriction imposed on the $s_{p\nu}$ carries with it the analogous restriction for the t_ν, proving thereby that the right side of (6) is a unique representation of real numbers to the base n^n. That is, we demonstrate that $t_\nu < n^n - 1$ for infinitely many ν, unless $t_\nu = n^n - 1$ for all ν.

The specified domain of the $s_{p\nu}$ is $0 \leqq s_{p\nu} \leqq n-1$; accordingly we have

$$0 \leqq \sum_{1 \leq p \leq n} s_{p\nu} \cdot n^{n-p} \leqq (n-1) \sum_{1 \leq p \leq n} n^{n-p} = n^n - 1. \tag{8}$$

Since $s_{p\nu} < n-1$ for infinitely many ν,

$$t_\nu = \sum_{1 \leq p \leq n} s_{p\nu} \cdot n^{n-p} < n^n - 1$$

for infinitely many ν, unless $s_{p\nu} = n-1$ for all values of p and ν.

To complete the proof of (i) it remains, therefore, only to show that the correspondence

$$(x_1, \cdots, x_n) \rightarrow \sum_{1 \leq p \leq n} h^p(x_p) \tag{9}$$

is one-to-one. We demonstrate, namely, that the right side of (6) determines a unique point in E^n:

Let

$$\sum_{1 \leq p \leq n} h^p(y_p) = n^{1-n} \sum_{1 \leq \nu \leq \infty} t'_\nu \cdot n^{-n\nu};$$

if

$$\sum_{1 \leq p \leq n} h^p(x_p) = \sum_{1 \leq p \leq n} h^p(y_p), \tag{10}$$

then $t_\nu = t'_\nu$ for all values of ν.

By the definition of the summands in (10), this equation is equivalent to the statement that

$$t_1 - t'_1 = \sum_{2 \leq \nu \leq \infty} (t'_\nu - t_\nu) \cdot n^{n-n\nu} = \alpha,$$

where $|\alpha| \leqq 1$, as shown with a simple calculation. That this inequality is strict follows from the fact that $|\alpha| = 1$ if and only if $|t'_\nu - t_\nu| = n^n - 1$ for all $\nu \geqq 2$, and then, according to the normalizing restriction, $|t'_1 - t_1| = n^n - 1 = 1$. The last equality is clearly impossible.

We now prove the assertion made, that $t_\nu = t'_\nu$ for all ν, by induction on ν. Since $|\alpha| < 1$ and the difference $|t'_1 - t_1|$ is integral or zero, it follows that $\alpha = 0$ and, hence, $t_1 = t_1'$. Suppose now that $t_\nu = t_\nu'$ for all $\nu \leqq k$, where $k \geqq 1$. Then equation (10) is equivalent to the statement that

$$t_{k+1} - t'_{k+1} = \sum_{k+2 \leq \nu \leq \infty} (t'_\nu - t_\nu) \cdot n^{n(k+1-\nu)} = \alpha',$$

where $|\alpha'| \leqq 1$. The above reasoning shows that inevitably $t_{k+1} = t'_{k+1}$, thereby completing the induction.

Now let

$$t'_\nu = \sum_{1 \leq p \leq n} s'_{p\nu} \cdot n^{n-p};$$

the relation $t_\nu = t_\nu'$ permits us to write

$$0 \leqq s_{n\nu} = s'_{n\nu} + \left[n \cdot \sum_{1 \leq p \leq n-1} (s'_{p\nu} - s_{p\nu}) \cdot n^{n-p-1} \right] \leqq n - 1.$$

Since $s'_{n\nu}$ is non-negative, and the expression in brackets is an integral multiple of n, this is impossible, unless $s_{p\nu} = s'_{p\nu}$ for all admitted values of p. This shows that $t_\nu \neq t_\nu'$ unless $s_{p\nu} = s'_{p\nu}$ for all p, and hence the correspondence (9) is one-to-one.

The correspondence (9) maps the unit cube, E^n, onto the closed interval $[0, n^{1-n}]$, in a one-to-one manner. To each point

$$y = \sum_{i \leq p \leq n} h^p(x_p)$$

in this interval, the assignment $g(y) = f(x_1, \cdots, x_n)$, therefore, is defined uniquely. The function g is that demanded in part (ii) of our theorem.

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