A NEW PROOF OF DEICKE'S THEOREM ON HOMOGENEOUS FUNCTIONS

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We denote by R_n the n-dimensional number space of points $\{x^1, x^2, \cdots, x^n\}$, where the x^i are real numbers, and we use R_n' to denote R_n with the point $\{0, 0, \cdots, 0\}$ removed. Let L be a positive function of class C^4 defined on R_n' and positively homogeneous of degree one. Then, introducing the matrix g of elements

$$g_{ij} = \frac{\partial^2(\frac{1}{2}L^2)}{\partial x^i \partial x^j},$$

we give a new proof of the following theorem, due originally to A. Deicke [1].

THEOREM. *Let* $\det g$ *be constant on* R_n'. *Then* g *is constant on* R_n'.

It is known that the assumptions made imply that the matrix g is positive definite [1]. We first prove

LEMMA 1. *Let* x, y *be any two points in* R_n'. *Then* $\operatorname{Tr} g^{-1}(x)g(y) \geqq n$.

PROOF. Since the matrices $g(x)$, $g(y)$ are positive definite, the characteristic roots of $g(y)$ with respect to $g(x)$ are all positive. These roots are also the characteristic roots of the matrix $g^{-1}(x)g(y)$ so that, using the inequality between arithmetic and geometric means,

$$\operatorname{Tr} g^{-1}(x)g(y) \geqq n(\det g^{-1}(x)g(y))^{1/n} = n.$$

We next introduce the elliptic differential operator

$$\Delta = \sum_{i,j=1}^{n} g^{ij} \frac{\partial^2}{\partial x^i \partial x^j},$$

where g^{ij} denotes the general element of the matrix $g^{-1}(x)$ and prove

LEMMA 2. *The matrix* Δg *is positive semi-definite.*

PROOF. Define a function ϕ_x by $\phi_x(y) = \operatorname{Tr} g^{-1}(x)g(y)$. Since $\phi_x(x) = n$, Lemma 1 shows that ϕ_x has a minimum at $y = x$ and hence the matrix of elements

Received by the editors September 28, 1963.

[1] This research was supported by the National Science Foundation (G-24154).

$$\frac{\partial^2 \phi_x}{\partial y^h \partial y^k}$$

is positive semi-definite for $y=x$. This matrix is also equal to Δg for $y=x$.

We complete the proof of the theorem by using a theorem due to E. Hopf [2, Theorem 2.1]. Lemma 2 implies that, for each h, $\Delta g_{hh} \geqq 0$. Since g_{hh} is positively homogeneous of degree zero and hence attains a maximum on R_n', Hopf's theorem shows that g_{hh} is constant on R_n'. Lemma 2 now implies that $\Delta g_{hk}=0$ for all h, k and, as before, Hopf's theorem shows that g_{hk} is constant on R_n'.

The author wishes to thank Professor H. C. Wang for a simplification to a previous proof.

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