HOLOMORPHIC IMPRIMITIVITIES

PLATON C. DELIYANNIS

0. Introduction. The well-known imprimitivity theorem of G. W. Mackey (see [1], [2]) provides a necessary and sufficient condition that a representation U of a group G on a Hilbert space is induced, i.e., realized by means of the action of G on vector-valued functions on a quotient space of G ; the condition can be formulated essentially as the requirement that a commutative C^* -algebra A of operators on the representation space is normalized by G (that is $U(\sigma)AU(\sigma^{-1}) = A$ for all $\sigma \in G$).

There is, however, a class of representations realized in a similar manner by the action of G on vector-valued holomorphic functions for which Mackey's theorem does not apply. The purpose of this paper is to define exactly the process of holomorphic induction and supply an analog of the imprimitivity theorem. Now R. J. Blattner has defined and studied in [3] a process of infinitesimal induction which, in certain cases, yields the same result as ours; one important difference is that he considers only functions with values in finite-dimensional spaces, although he does not require them to be holomorphic in all variables; further, he does not discuss an imprimitivity theorem.

As an application of our results, one can explain why the representations, say, of the discrete series of the 2×2 real unimodular group have the form given by Bargmann in [4].

1. Hilbert spaces of holomorphic functions. Let Z be a complex manifold and let $d\mu$ be a finite measure on Z . Suppose that K is a Hilbert space and consider the collection of all functions $f: Z \rightarrow K$ which are holomorphic and are in $L_2(d\mu)$; clearly, this is a subspace of $L_2(d\mu)$, and, in fact, a closed subspace of it. The argument for this follows that of the scalar case, and is based upon the inequality $\|f(z)\| \leq \Theta(z, z)^{1/2} \|f\|_2$, where Θ is the reproducing kernel of the manifold Z relative to $d\mu$ (whose existence we assume). So we have obtained a Hilbert space of holomorphic vector-valued functions, which we shall write as H .

It is clear that for any bounded holomorphic scalar function ξ on Z , we have that $\xi f \in H$, provided $f \in H$. We have in this way an algebra of bounded operators on H which is obviously commutative and antisymmetric. We shall assume that the manifold Z admits suffi-

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ciently many bounded scalar holomorphic functions so that they separate points. In that case the algebra of all finite sums $\sum \xi_i \bar{f}_i$ (with $\xi_i, \bar{f}_i$ bounded holomorphic) will be uniformly dense in the continuous bounded functions on Z .

2. Representations and holomorphic imprimitivities. Let us now assume that for each σ in some locally compact group G of holomorphic homeomorphisms of Z , and for each $z \in Z$, a bounded operator $m(\sigma, z)$ on K is defined, having the properties:

M1: For each $z \in Z$, $\sigma \rightarrow m(\sigma, \sigma z)$ is strongly continuous and, for each $\sigma \in G$, $z \rightarrow m(\sigma, z)$ is holomorphic.

M2: $m(\sigma\tau, z) = m(\sigma, z)m(\tau, \sigma^{-1}z)$.

M3: $m(\sigma, z)^* m(\sigma, z) = (d(\sigma\mu)/d\mu)(z)I$, where $d(\sigma\mu)$ is defined by the formula $\int_Z F(z) d(\sigma\mu)(z) = \int_Z F(\sigma z) d\mu(z)$, and where σz is the result of the action of the group element σ on the point $z \in Z$. We shall assume this action to be jointly continuous.

We shall call such an object a holomorphic multiplier.

It is clear that for each $\sigma \in G$ the function $\sigma \cdot f: z \rightarrow m(\sigma, z)f(\sigma^{-1}z)$ is holomorphic whenever f is, and is anyway defined. Because of M2, we have that $(\sigma\tau) \cdot f = \sigma \cdot (\tau \cdot f)$, while M3 implies that $\int_Z \langle \sigma \cdot f, \sigma \cdot g \rangle d\mu = \int_Z \langle f, g \rangle d\mu$ for any σ and any $f, g \in L_2(d\mu)$ with values in K (we have denoted the inner product in K by $\langle \cdot, \cdot \rangle$). Finally, M1 implies that the functions $\sigma \rightarrow \int_Z \langle \sigma \cdot f, g \rangle d\mu$ are continuous for any $f, g \in L_2$, hence for any $f, g \in H$. It is clear that if we denote the operator $f \rightarrow \sigma \cdot f$ by $U(\sigma)$, then U is a continuous unitary representation of G on $L_2(d\mu)$, and since H is a closed invariant subspace, the restriction of U to H is also a representation of G . We shall denote this again by U and call it the representation induced by the multiplier m .

A simple computation shows that for any scalar bounded holomorphic function ξ on Z , we have $U(\sigma)\xi U(\sigma^{-1}) = \sigma\xi$, where $(\sigma\xi)(z) = \xi(\sigma^{-1}z)$. We shall describe this situation by saying that the algebra of scalar bounded holomorphic functions on H forms a holomorphic imprimitivity. Note that this algebra has the following two properties, which can be verified immediately:

I1: For any $f, g \in H$ the map $\sum_{i=1}^n \xi_i \bar{f}_i \rightarrow \sum_{i=1}^n \langle \xi_i f, \bar{f}_i g \rangle$ is the restriction of a unique bounded measure on Z .

I2: There exists a closed subspace L of H such that the collection of all vectors $\sum \xi_i l_i$ is dense in H while the ratio $\langle \xi l_1, \bar{\xi} l_2 \rangle / \langle l_1, l_2 \rangle$ does not depend on $l_1, l_2 \in L$. (L consists of all constant functions on Z .)

Comments. (i) Condition M2 is both necessary and sufficient for the relation $(\sigma\tau) \cdot f = \sigma \cdot (\tau \cdot f)$, since the set of all vectors $f(z)$ with z fixed and f variable is dense in K .

(ii) Condition M3 is also necessary and sufficient for the action of U to be unitary, because of the density of all functions $\sum \xi_i \bar{\xi}_i$.

3. Conditions for holomorphic induction. Let U be a continuous unitary representation of the group G on a Hilbert space H and assume that:

G1: The group G is a locally compact transformation group of holomorphic homeomorphisms of a complex manifold Z .

G2: There exists an algebra C of bounded operators ξ on H which is a holomorphic imprimitivity, i.e., is algebraically isomorphic to an algebra of bounded scalar holomorphic functions on Z , and, further, conditions I1, I2 of §2 are satisfied.

G3: The finite measure $d\mu$ on Z which, according to I1, I2 has the property that $\langle \xi l_1, \zeta l_2 \rangle = \langle l_1, l_2 \rangle \int \xi \bar{\zeta} d\mu$, determines a reproducing kernel Θ on Z such that, for each $w \in Z$, the function $z \rightarrow \Theta(z, w)$ is bounded; we shall denote this function by $\hat{\Theta}(w)$. Finally, suppose that the holomorphic functions ξ are dense in the holomorphic part of $L(d\mu)$.

Under these conditions we shall show that the representation U is holomorphically induced by some multiplier.

Comments. (i) Given an algebra of operators, there are always subspaces for which $\langle \xi l_1, \zeta l_2 \rangle / \langle l_1, l_2 \rangle$ is independent of l_1, l_2 , and, in fact, maximal such subspaces. So that all that I2 actually asserts is that some such maximal subspace has a complement invariant under the algebra.

(ii) Condition I1 is indispensable in the sense that there can be constructed hermitian functionals $B(\xi, \zeta)$ such that $|B(\xi, \zeta)| \leq \|\xi\|_\infty \|\zeta\|_\infty$, but still they are not the restriction of any measure.

(iii) Condition G3 is actually the only one which is not needed in the construction of the representation from a multiplier.

4. Analysis of the vectors. Let $z \in Z$ and consider the map $l \rightarrow \langle \hat{\Theta}(z)l, f \rangle$, with $l \in L$ and $f \in H$. Since this is a bounded functional on L , there is a unique $f(z) \in L$ such that $\langle \hat{\Theta}(z)l, f \rangle = \langle l, f(z) \rangle$ for all $l \in L$.

LEMMA (4.1). *If $f = \sum_{i=1}^n \xi_i l_i$, then $f(z) = \sum_{i=1}^n \xi_i(z) l_i$, and, hence, f is holomorphic.*

PROOF. We have $\langle \hat{\Theta}(z)l, \sum \xi_i l_i \rangle = \sum \langle \hat{\Theta}(z)l, \xi_i l_i \rangle = \sum \langle l, l_i \rangle \int \hat{\Theta}(z) \xi_i d\mu = \sum \langle l, l_i \rangle \int \Theta(w, z) [\xi_i(w)]^- d\mu(w) = \sum \langle l, l_i \rangle [\xi_i(z)]^- = \sum \langle l, \xi_i(z) l_i \rangle$.

LEMMA (4.2). *For any $f = \sum \xi_i l_i$, $g = \sum \zeta_j k_j$, we have that $\langle f, g \rangle = \int \langle f(z), g(z) \rangle d\mu(z)$.*

PROOF. Because we have

$$\begin{aligned}\langle f, g \rangle &= \sum \langle \xi_i l_i, \zeta_j k_j \rangle = \sum \langle l_i, k_j \rangle \int \xi_i \bar{\zeta}_j d\mu \\ &= \int \langle \sum \xi_i(z) l_i, \sum \zeta_j(z) k_j \rangle d\mu(z) = \int \langle f(z), g(z) \rangle d\mu(z).\end{aligned}$$

So, if we call H_0 the dense subspace of all sums $\sum \xi_i l_i$, we have that the map T defined at the beginning of this section transforms H_0 unitarily into the L_2 space relative to $d\mu$, with values in the Hilbert space L . But the image of H_0 consists of all functions $\sum \xi_i(z) l_i$, which are dense in the holomorphic part of L_2 . Since we have that convergence in the norm in L_2 implies pointwise convergence (in fact, uniform convergence on compact sets), we have shown:

THEOREM (4.1). *The map T is a unitary map between H and the holomorphic part of $L_2(d\mu)$ over Z ; $T(\xi f)(z) = \xi(z)(Tf)(z)$.*

5. Action of the group. Now for each $z \in Z$ and each $\sigma \in G$, there is defined an operator $m(\sigma, z)$ on L by means of $l \rightarrow (U(\sigma)l)(z)$.

THEOREM (5.1). *This correspondence determines a holomorphic multiplier on Z . For any $f \in H$ we have that $U(\sigma)f: z \rightarrow m(\sigma, z)f(\sigma^{-1}z)$, i.e., U is induced by m .*

PROOF. We show this last relation first; again, it suffices to show this for a dense subspace because $m(\sigma, z)$ is bounded. So, let $f = \sum \xi_i l_i$; then $U(\sigma)f = \sum U(\sigma)\xi_i l_i = \sum U(\sigma)\xi_i U(\sigma^{-1})U(\sigma)l_i = \sum (\sigma\xi_i)(U(\sigma)l_i)$, and, evaluating both members at z , we obtain $(U(\sigma)f)(z) = \sum \xi_i(\sigma^{-1}z)m(\sigma, z)l_i = m(\sigma, z) \sum \xi_i(\sigma^{-1}z)l_i = m(\sigma, z)f(\sigma^{-1}z)$. To show M1 we note that $\sigma \rightarrow m(\sigma, \sigma z) = (U(\sigma)f)(z)$ is the composite of two continuous functions, hence continuous on G . It is clear that M2 follows from the assumption that U is a representation. Unitarity of U implies that

$$\begin{aligned}\int (\sum \xi_i(z) [\zeta_i(z)]^{-1}) \langle f(z), g(z) \rangle d\mu(z) \\ = \int (\sum \xi_i(z) \zeta_i(z)) \langle m(\sigma, \sigma z)^* m(\sigma, \sigma z) f(z), g(z) \rangle d(\sigma^{-1}\mu)(z)\end{aligned}$$

for any $f, g, \sigma, \xi_i, \zeta_i$; but then we get that the two measures $\langle f(z), g(z) \rangle d\mu(z)$ and $\langle m(\sigma, \sigma z)^* m(\sigma, \sigma z) f(z), g(z) \rangle d(\sigma^{-1}\mu)(z)$ are identical, and from this we have M3 immediately.

6. An application. We shall show that, in the case of the group G of all 2×2 real unimodular matrices, and with U an irreducible representation $D^+(k)$ of the discrete series, such a holomorphic imprimitivity exists by virtue of the structure of the associated infinitesimal

representation of the Lie algebra of G . Notations and results shall be drawn from [4], and we shall omit all computations.

Now the element $\sigma \in G$ can be realized as a matrix

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

with a, b complex numbers satisfying $|a|^2 - |b|^2 = 1$, acting transitively on the unit disc Z by $z \rightarrow \sigma z = (\bar{a}z + \bar{b})/(bz + a)$; condition G1 is clear. The Lie algebra of G is spanned by

$$\chi_0 = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \chi_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \text{and} \quad \chi_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which are mapped by the infinitesimal representation associated to U onto H_0, H_1 and H_2 , respectively. Bargmann has shown in [4] that, in case the operator $Q = H_1^2 + H_2^2 - H_0^2 = qI$ is given by $q = k(1 - k)$ with $k = l/2 > 0$ (l an integer), then there exists an orthonormal basis $f_k, f_{k+1}, \dots$ in the representation space such that: (i) $H_0 f_{k+s} = (k+s)f_{k+s}$, (ii) $Ff_{k+s} = \rho_{s+1}^{1/2} f_{k+s+1}$, $Gf_{k+s} = \rho_s^{1/2} f_{k+s-1}$ ($Gf_k = 0$) where $F = H_1 + iH_2$, $G = H_1 - iH_2$ and $\rho_s = q + (k+s-1)(k+s)$.

We define an operator z on the representation space by means of $f_{k+s} \rightarrow -(s+1)\rho_{s+1}^{1/2} f_{k+s+1}$, and observe that its bound is ≤ 1 . Furthermore we have immediately that $zG = k - H_0$ on any finite linear combination of the f_{k+s} . Now we see that any power series $\xi(z) = \sum a_n z^n$ will act on our representation space provided that $\sum |a_n| < \infty$; this produces an isomorphism between the algebra of all holomorphic functions on Z represented by such power series and the corresponding operators. To verify G2 we observe that $z^n f_k$ is a multiple of f_{k+n} so that we can take L to be the space spanned by $\{f_k\}$; then $\langle \xi l_1, \xi l_2 \rangle / \langle l_1, l_2 \rangle$ is obviously independent of l_1, l_2 while $\langle z^n f_k, z^m f_k \rangle = (n!m! / r_n r_m) \delta_{nm}$, where $r_n = \prod_{s=1}^n \rho_s^{1/2}$; but this is proportional to $\int_Z (1 - z\bar{z})^{l-2} z^n \bar{z}^m dx dy$, which means that I1, I2 and therefore G2 are satisfied. It is well known that the measure $d\mu(z) = (1 - z\bar{z})^{l-2} dx dy$ has a reproducing kernel with the required properties, and thus G3 becomes obvious. Finally we check the condition $U(\sigma)\xi U(\sigma^{-1}) = \sigma\xi$; it suffices to do this for $\xi(z) = z$. Now the action of G on H_i is known (see [4]), and, using this as well as the commutation relations for H_0, F, G and the relation $zG = k - H_0$, we obtain that on every finite linear combination of the f_{k+s} we have $U(\sigma)zU(\sigma^{-1})(\bar{a} - bz)(bH_0 + bk + \bar{a}G) = (az - \bar{b})(bH_0 + bk + \bar{a}G)$. But the range of $bH_0 + bk + \bar{a}G$ is dense in the space, as can be seen from (ii) above; so we have

$$U(\sigma)zU(\sigma^{-1})(\bar{a} - bz) = (az - \bar{b}):$$

the power series for $(\bar{a} - bz)^{-1}$, however, converges as previously required, hence we have that $U(\sigma)zU(\sigma^{-1}) = (az - \bar{b})/(\bar{a} - bz) = \sigma^{-1}z$.

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ILLINOIS INSTITUTE OF TECHNOLOGY

ON WANDERING SUBSPACES FOR UNITARY OPERATORS

J. B. ROBERTSON¹

Let V be a unitary operator on a complex Hilbert space H . X is said to be a wandering subspace for V if it is a subspace of H such that $V^m(X) \perp V^n(X)$ for all $m \neq n$. The purpose of this note is to study the relation between two wandering subspaces X and Y satisfying $\sum_{k=-\infty}^{\infty} V^k(X) \subseteq \sum_{k=-\infty}^{\infty} V^k(Y)$.

THEOREM 1. *Let X and Y be wandering subspaces for a unitary operator V such that:*

- (a) $\sum_{k=-\infty}^{\infty} V^k(X) \subseteq \sum_{k=-\infty}^{\infty} V^k(Y)$,
- (b) $\dim(X) = \dim(Y) < \infty$.

Then $\sum_{k=-\infty}^{\infty} V^k(X) = \sum_{k=-\infty}^{\infty} V^k(Y)$.

PROOF. Let $x_1, \dots, x_n$ and $y_1, \dots, y_n$ be orthonormal bases for X and Y , respectively. Since $x_i \in \sum_{k=-\infty}^{\infty} V^k(Y)$ we have

$$x_i = \sum_{r=1}^n \sum_{k=-\infty}^{\infty} a_{ir k} V^k(y_r), \quad a_{ir k} = (x_i, V^k(y_r)),$$

$$\sum_{r=1}^n \sum_{k=-\infty}^{\infty} |a_{ir k}|^2 < \infty, \quad i = 1, \dots, n.$$

It follows that

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