THE COLLINEATION GROUPS OF FREE PLANES. II: A PRESENTATION FOR THE GROUP G_2

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In [2], I exhibited a set of (three) generators for the collineation group, G_2 , of the free plane π^2 . In the last section of that paper, I claimed to have a set of relations for the group, but had omitted them due to the length of the proof that they sufficed to determine the group. Several requests for the relations (and my own realization that this particular group could not easily be studied by the purely geometric presentation of [2]) led to the preparation of the present paper, in which the proof of the sufficiency of the relations is somewhat different from the original proof. I would like to thank Professor W. H. Mills for several helpful discussions concerning the present proof.

This paper was written as an extension of [2], and the notation and terminology are exactly the same as in [2] except in one instance, and the difference is carefully explained when the subject is introduced.

In studying the group G_2 , it seems necessary to make heavy use of the geometric properties of π^2 , so we begin with an examination of certain relations between the generating configurations in π^2 . As in [2], we have π_0^2 —a set of four points, no three of which are collinear—which generates π^2 , and relative to which we assign a stage to every point and line of π^2 . A generating configuration, C (consisting of four points of π^2 no three collinear, which generate π^2), is said to be of stage k if each of the four points of C has stage $\leq k$, and if some point of C has stage k . We have the following result.

THEOREM A. *There is, in π^2 , one generating configuration of stage 0, and exactly $6 \cdot 4^{k-1}$ configurations of stage $k > 0$.*

PROOF. This theorem is easily proved using the proof of Theorem 6 in [2]. The configuration π_0^2 is the (unique) configuration of stage 0, and it was observed in [2, §3] that there are exactly seven configurations of stage ≤ 1 . Thus the theorem is true for $k = 1$. For $k > 1$, we examine Figure 1 of [2]. The configuration $C = \{P_1P_2P_3P_4\}$ was a general configuration of stage $k > 1$. In the course of the proof of Theorem 6, it was shown that in order for C to be a generating con-

Presented to the Society, January 17, 1962; received by the editors, November 14, 1963.

figuration, two of its diagonal points, say d_1 and d_2 , must have stage $< k$, while the third point, $(P_1 \cdot P_3) \cap (P_2 \cdot P_4) = d_3$ must have stage $\geq k$. But, in fact, d_3 must have stage $k+1$, for if the stage of d_3 were k , both lines $P_1 \cdot P_3$ and $P_2 \cdot P_4$ must have stage $\leq k$ (recall that any point of stage k is on exactly two lines of stage $\leq k$, at least one of which has stage k) and we could conclude that the stages of P_1, P_2, P_3, P_4 are all $< k$, a contradiction. Thus, of the six generating configurations defined by C and d_1, d_2, d_3 , exactly two, $\{d_1, d_2, P_1, P_3\}$ and $\{d_1, d_2, P_2, P_4\}$ have stage $\leq k$, while the other four have stage $k+1$. Observe now that two distinct quadrangles of stage $\leq k$ cannot generate in the above manner the same quadrangle of stage $k+1$, and we have shown that each configuration of stage $k \geq 1$ generates exactly four others of stage $k+1$. The proof is now completed by induction on k .

We next define the *stage of a collineation* α as the stage of the configuration, $(\pi_0^2)\alpha$, which is the image of π_0^2 under α . As an immediate corollary to Theorem A, we have:

THEOREM B. *There are exactly 24 collineations of stage 0, and*

$$(1) \quad f(k) \equiv 24 \cdot 6 \cdot 4^{k-1}$$

collineations of stage $k \geq 1$.

PROOF. As we saw in [2] (Theorem 4), there are exactly 24 collineations which map π_0^2 onto any other generating configuration of π^2 .

Recall now the set of generators obtained in [2] for G_2 . If $\pi_0^2 = \{A_1 A_2 B_1 B_2\}$, and if $a_1 = (A_1 \cdot A_2) \cap (B_1 \cdot B_2)$, $a_2 = (A_1 \cdot B_2) \cap (A_2 \cdot B_1)$, then it was shown that the collineations determined by

$$\theta_1 = \begin{Bmatrix} A_1 \rightarrow A_2 \\ A_2 \rightarrow A_1 \\ B_1 \rightarrow B_1 \\ B_2 \rightarrow B_2 \end{Bmatrix}, \quad \theta_2 = \begin{Bmatrix} A_1 \rightarrow A_2 \\ A_2 \rightarrow B_1 \\ B_1 \rightarrow B_2 \\ B_2 \rightarrow A_1 \end{Bmatrix}, \quad \phi = \begin{Bmatrix} A_1 \rightarrow A_1 \\ A_2 \rightarrow a_1 \\ B_1 \rightarrow B_1 \\ B_2 \rightarrow a_2 \end{Bmatrix}$$

generate G_2 , where θ_1 and θ_2 generate a group, S^2 , isomorphic with the symmetric group on the four points of π_0^2 , while $\phi^2 = I$. Thus, any element $\alpha \in G_2$ can be written

$$(2) \quad \alpha = P_{i_1} \phi P_{i_2} \phi \cdots P_{i_n} \phi P_{i_{n+1}},$$

where $P_{i_k} \in S^2$. We shall say that α is a group element of *length* n if α can be written in the form (2) containing n occurrences of ϕ . The next result we can prove is

THEOREM C. *Every collineation of stage n is of length $\leq n$.*

PROOF. The proof of this theorem is essentially contained in the proof of Theorem 7 of [2]. For $n=0$, the 24 collineations of stage 0 are the elements of S^2 , and they are of length 0. The $24 \cdot 6$ elements of stage 1 are obtained from the six collineations of length 1 listed on p. 135 of [2] by premultiplying each by the elements of S^2 . For $n > 1$, let α be the collineation of stage n under consideration. Then it was demonstrated in the proof of Theorem 7 that there exist collineations β, η , such that β is of stage $k < n$, and η is of stage ≤ 1 , and such that $\alpha = \eta\beta$. Now if we assume as an induction hypothesis that the theorem is true for all $k < n$, then, since β is assumed to be of length $< n$, and η has length ≤ 1 , the length of α must be no greater than $k+1 \leq n$, and the theorem is proved.

With these theorems at our disposal, we can proceed to derive the relations defining G_2 . This is done by considering the number of elements of G_2 of each possible length. Firstly, there are exactly 24 elements of length 0, the elements of S^2 . These are the collineations of stage 0. The collineations of length 1 are not quite so simple to determine. There are 24^2 possibilities for collineations of length 1, namely all $P_{i_1}\phi P_{i_2}$, $P_{i_1}, P_{i_2} \in S^2$. But these different words can represent 24^2 distinct collineations only if no relations of the type

$$(3) \quad \phi P_{i_1} = P_{i_2} \phi$$

are satisfied for $P_{i_1}, P_{i_2} \in S^2$ except for $P_{i_1} = P_{i_2} = I$. But one can readily verify by computation the validity of

$$(3a) \quad \phi P_6 = P_7 \phi,$$

$$(3b) \quad \phi P_7 = P_6 \phi,$$

$$(3c) \quad \phi P_8 = P_8 \phi,$$

where

$$(4) \quad P_6 = (A_1 B_1)(A_2)(B_2), \quad P_7 = (A_1 B_1)(A_2 B_2), \quad P_8 = (A_2 B_2)(A_1)(B_1).$$

(Here, the more cumbersome notation of [2] for collineations has been abandoned in favor of standard permutation notation, since from this point to the end of the paper, all collineations to be considered are in S^2 , and are representable by their action on π_0^2 .) Observe that the set $P_0 = I, P_6, P_7, P_8$ forms a subgroup, S' , of order four, of the group S^2 . Now, a set of coset representatives for the left cosets of S' in S^2 is $S = \{P_0, P_1, P_2, P_3, P_4, P_5\}$, where

$$(5) \quad \begin{aligned} P_1 &= (A_1 A_2)(B_1)(B_2), \quad P_2 = (A_1 B_2)(A_2)(B_1), \quad P_3 = (B_1 A_2)(A_1)(B_2), \\ P_4 &= (B_1 B_2)(A_1)(A_2), \quad P_5 = (A_1 A_2)(B_1 B_2). \end{aligned}$$

It is now easy to see that any element of length 1 can be written in the form

$$(6) \quad P_r \phi P_k, \quad P_r \in S^2, \quad P_k \in S.$$

For, if $P_i, P_j \in S^2$, $P_i \phi P_j = P_i \phi P_s P_k$, $P_s \in S'$, $P_k \in S$, and $P_i \phi P_s P_k = P_i P_i \phi P_k = P_r \phi P_k$, $P_i \in S'$. Thus, the relations (3) imply that there can be at most $24 \cdot 6 = f(1)$ elements of length 1. But Theorems B and C (and the fact that all the collineations of length 0 are of stage 0) imply that there must be at least $f(1)$ collineations of length 1. Thus the collineations determined by (6) *must all be distinct*, and must all correspond to the collineations of stage 1.

We now need to verify the equality

$$(7) \quad P_5 \phi P_5 = \phi P_5 \phi$$

which is easily done by direct computation, and we can proceed to prove

THEOREM D. *The relations determined by (3) and (7) imply that there can be no more than $f(k)$ collineations of length $k > 1$.*

PROOF. The proof is straightforward and proceeds as follows. For $k=2$, let $g = P_{i_1} \phi P_{i_2} \phi P_{i_3}$ be any element of length 2. Then we can write $P_{i_2} \phi P_{i_3} = P_{j_2} \phi P_n$ for some $P_n \in S$. Similarly, we have $P_{i_1} \phi P_{j_2} = P_i \phi P_m$ for some $P_m \in S$. Thus, $g = P_i \phi P_m \phi P_n$, $P_m, P_n \in S$. But if $P_m = I$ or $P_m = P_5$, g would be of length < 2 . Thus, we can assert, for any g of length 2,

$$(8) \quad g = P_i \phi P_m \phi P_n, \quad P_i \in S^2, \quad P_n \in S, \quad P_m \in S - \{P_0, P_5\}.$$

In general, we can show for any element g of length k ,

$$(8') \quad g = P_i \phi P_m \phi \cdots P_{m_{k-1}} \phi P_n, \quad P_i \in S^2, \quad P_{m_j} \in S - \{P_0, P_5\}, \quad P_n \in S.$$

Now, since there are exactly four elements in $S - \{P_0, P_5\}$, there can be no more than $24 \cdot 4^{k-1} \cdot 6 = f(k)$ such elements of length k .

This last theorem can be combined with the previous theorems to prove

THEOREM E. (a) *Every collineation of stage n is of length n ;* (b) *The relations (3) and (7) imply that there are exactly $f(k)$ elements of length k .*

PROOF. Part (b) of the theorem follows from the first part (which is an improvement on Theorem C), for if each collineation of stage k

is of length k , there must be, by Theorem B, exactly $f(k)$ collineations of length k . But Theorem D states that relations (3) and (7) show there must be $\leq f(k)$ collineations of length k . Thus, since *no* relations can hold in G_2 which imply that there are less than $f(k)$ such collineations, the above-mentioned relations must indeed imply that Theorem D can be sharpened, as asserted in part (b) of the theorem. To prove (a) we apply an induction argument. It has already been shown that every collineation of stage 0 is of length 0 and every collineation of stage 1 is of length 1. Assume that every collineation of stage k is of length k for $k < n$. Then, for $k < n$, there must be exactly $f(k)$ collineations of length k . Now since by Theorem C, every collineation of stage n is of length $\leq n$, and since by Theorem D there are $\leq f(n)$ collineations of length n , there must be exactly $f(n)$ collineations of length n , *and they must all be of stage n* , since all of the collineations of length $< n$ are also of stage $< n$. Thus, every collineation of stage n is of length n , and we have concluded the proof of the theorem.

Any relations in G_2 not implied by (3) and (7) (and the fact that θ_1 and θ_2 generate a group isomorphic with S_4), must reduce the number of elements of some length. But by Theorem E, these relations already have reduced the number of words of a given length in the free product of S^4 and ϕ to the required number, so any further relations would identify nonequal elements of G_2 . Thus we have obtained a set of relations for G_2 . To simplify further, notice that S' is the *four group*. Thus, if we have (3a), we can write

$$(\phi P_6)(\phi P_6) = (\phi P_6)(P_7\phi) = \phi P_8\phi = (P_7\phi)(\phi P_6) = P_8,$$

which implies $\phi P_8 = P_8\phi$, or (3c). Also from (3a), we can write

$$(\phi P_6)^{-1} = P_6\phi = (P_7\phi)^{-1} = \phi P_7,$$

which implies (3b).

We summarize our results in

THEOREM F. *The group G_2 is generated by θ_1 , θ_2 and ϕ , and the following relations suffice to define the group:*

(a) *Those relations insuring that θ_1 and θ_2 generate S_4 , the symmetric group on $\{A_1, A_2, B_1, B_2\}$: $\theta_1^2 = \theta_2^4 = (\theta_1\theta_2)^3 = I$.*

(b) $\phi^2 = I$,

(c) $\phi\theta_2\theta_1\theta_2^2\theta_1 = \theta_2^2\phi$,

(d) $[(\theta_2^2\theta_1)^2\phi]^3 = I$.

PROOF. It suffices to apply Theorem E and check the equalities

$$(9) \quad \left\{ \begin{array}{l} (\theta_2^2 \theta_1)^2 = P_6 \\ \theta_2^2 = P_7 \\ \theta_2^{-1} P_6 = P_6 \end{array} \right\}.$$

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PSEUDO-ISOTOPICALLY CONTRACTIBLE SPACES

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The space M is called pseudo-isotopically contractible provided that if A is a compact subset of M there is a continuous function $r(x, t)$ of $M \times [0, 1]$ into M such that (1) if $t < 1$, $r|_{M \times t}$ is a homeomorphism onto, and (2) if $t = 1$, $r|_{A \times 1}$ is a point.

Let X be a locally euclidean n -dimensional space with the property that each pair of points lies in the interior of some n -ball. Clearly X is a connected n -manifold without boundary.

THEOREM. *If M is a locally euclidean n -dimensional space with the property that each pair of points lies interior to some n -ball, then M is an open n -cell if and only if M is pseudo-isotopically contractible.*

LEMMA. *If M is pseudo-isotopically contractible and p is a point of M , the function $r(x, t)$ may be chosen so that $r(A, 1) = p$.*

PROOF. Let U be the interior of a ball containing p and q . Suppose U is given a co-ordinate system $(x_1, \dots, x_n)$, where $x_1^2 + \dots + x_n^2 < 1$ and $(x_1, \dots, x_n) \in \bar{U} \setminus U$ if and only if $x_1^2 + \dots + x_n^2 = 1$.

Let $0 < \epsilon < 1$; then the mapping

$$\begin{aligned} x'_1 &= x_1 + \epsilon t p, & 0 \leq t \leq 1, \\ x'_i &= x_i, & i > 1, \end{aligned}$$

Received by the editors November 19, 1963.