THE DERIVATIVE OF A MONOTONIC DISCONTINUOUS FUNCTION

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S. Marcus [1] recently showed that if E is a denumerable set of type G_δ on the x -axis, then there exists a nondecreasing function f whose derivative is infinite on E and zero outside the closure of E . He thus gave a partial answer to a question raised by Z. Zahorski [2, p. 52, Problem VII]. The present note extends the result of Marcus and thereby completes the solution of Zahorski's problem.

THEOREM. *If E is a denumerable set of type G_δ on the x -axis, then there exists a nondecreasing function f whose derivative satisfies the condition*

$$f'(x) = \begin{cases} \infty & (x \in E), \\ 0 & (x \notin E). \end{cases}$$

PROOF. Let $E = \{x_n\} (n=1, 2, \dots)$ be a set of type G_δ . Marcus used the function

$$f(x) = \sum 2^{-n} f_n(x),$$

where $f_n(x)$ takes the values 0, $1/2$, and 1 on the sets $x < x_n$, $x = x_n$, and $x > x_n$, respectively. We modify his construction by replacing the coefficients 2^{-n} with appropriate values c_n .

By hypothesis, $\{x_n\} = \bigcap E_j$ (E_j open, $E_j \supset E_{j+1}$). For each j , let $\{E(j, k)\}$ denote the family of components of E_j . Without loss of generality, we may suppose that each $E(j, k)$ has length less than 1.

Corresponding to the point x_1 we choose an index pair (j_1, k_1) such that $x_1 \in E(j_1, k_1)$; we denote by d_1 the distance between x_1 and the complement of $E(j_1, k_1)$; and we choose $c_1 = 2^{-1}d_1$. After the index pairs (j_m, k_m) have been determined for $m=1, 2, \dots, n-1$, we select a pair (j_n, k_n) different from all the pairs (j_m, k_m) ($1 \leq m < n$), in such a way that $x_n \in E(j_n, k_n)$; and we write $c_n = 2^{-n}d_n$, where d_n denotes the distance between x_n and the complement of $E(j_n, k_n)$. Since each x_n lies in infinitely many of the segments $E(j, k)$, the process can be continued without difficulty.

Now let $f(x) = \sum c_n f_n(x)$. Since f is nondecreasing and has a left-

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hand and a right-hand discontinuity at each point x_n , we see immediately that $f'(x) = \infty$ at each point of E .

On the other hand, let $x \notin E$, let M denote an integer such that $x \notin E(j_n, k_n)$ when $n > M$, and let $u \neq 0$. Clearly,

$$0 \leq \frac{f(x+u) - f(x)}{u} \leq |u|^{-1} \sum^* c_n,$$

where the sum $\sum^*$ extends over the terms with indices n for which

$$x - |u| \leq x_n \leq x + |u|.$$

If N denotes the least of these indices, then $N \rightarrow \infty$ as $u \rightarrow 0$. For all $N > M$, the terms c_n that enter the sum $\sum^*$ satisfy the condition $c_n = 2^{-n} d_n \leq 2^{-n} |u|$. Hence the difference quotient is less than 2^{1-N} , and hence $f'(x) = 0$. This completes the proof.

REFERENCES

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