THE MAXIMUM TERM OF AN ENTIRE SERIES WITH GAPS

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Let $\sum a_p z^p$ denote the power series for an entire function of order ρ and lower order λ. S. M. Shah [2] has shown that

$$\tag{1} \begin{aligned} \liminf_{r\to\infty} [\mu(r)]^{1/\nu(r)} &\leqq e^{1/\rho}, \\ \limsup_{r\to\infty} [\mu(r)]^{1/\nu(r)} &\geqq e^{1/\lambda}, \end{aligned}$$

where $\mu(r)$ denotes the maximum term of $\sum |a_p| r^p$ and $\nu(r)$ is the largest integer p for which $\mu(r) = |a_p| r^p$.

The object of the present note is to obtain a sharper form of (1) for those entire series which possess Hadamard gaps. For this purpose let the subsequence $\{a_{p_m}\}$ contain all the nonvanishing terms of $\{a_p\}$, and suppose that

$$\tag{2} \liminf_{m\to\infty} \frac{p_{m+1}}{p_m} \geqq 1 + \theta > 1.$$

We shall prove the following

THEOREM. *Suppose $\sum a_{p_m} z^{p_m}$ is an entire series of order ρ and lower order λ whose gaps satisfy* (2). *Then*

$$\tag{3} \begin{aligned} \liminf_{r\to\infty} [\mu(r)]^{1/\nu(r)} &\leqq \alpha^{1/\rho}, \\ \limsup_{r\to\infty} [\mu(r)]^{1/\nu(r)} &\geqq \beta^{1/\lambda}, \end{aligned}$$

where

$$\alpha = (1+\theta)^{1/\theta}$$

and

$$\beta = (1+\theta)^{(1+\theta)/\theta}.$$

We call attention to the fact that a series which satisfies (2) need not be of irregular growth; much larger gaps are needed [3] to insure that $\lambda < \rho$.

PROOF. The function $\nu(r)$ is a nondecreasing step function which is continuous from the right and assumes only nonnegative integer

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values. Therefore there is a nondecreasing sequence $\{R_k\}$ which $\nu(r)$ "counts," i.e.,

$$\nu(r) = \sum_{R_k \leq r} 1.$$

For convenience we assume that $R_1 = 1$. No generality is lost since this is equivalent to requiring that

$$|a_0| = \max_{p \geqq 1} |a_p|.$$

For each $k \geqq 1$, let

$$t_k = \log R_k - \frac{\log R_1 + \cdots + \log R_k}{k} \tag{4a}$$

and

$$u_k = \log R_{k+1} - \frac{\log R_1 + \cdots + \log R_k}{k}. \tag{4b}$$

In addition to satisfying (2), we assume the sequence

$$p_0, p_1, \cdots, p_m, \cdots$$

is such that $p_0 = 0$ and $p_1 = 1$. For notational convenience we shall always denote p_m by n.

The following relations are easily verified by examining the local minima and maxima of the quantities involved:

$$\liminf_{m\to\infty} t_n = \liminf_{r\to\infty} \frac{\log \mu(r)}{\nu(r)},$$

$$\limsup_{m\to\infty} u_n = \limsup_{r\to\infty} \frac{\log \mu(r)}{\nu(r)},$$

$$\liminf_{m\to\infty} \frac{\log R_n}{\log n} = \liminf_{r\to\infty} \frac{\log r}{\log \nu(r)} = \frac{1}{\rho},$$

$$\limsup_{m\to\infty} \frac{\log R_{n+1}}{\log n} = \limsup_{r\to\infty} \frac{\log r}{\log \nu(r)} = \frac{1}{\lambda}.$$

We shall also need estimates for the quantities

$$A(n) = 1 + \sum_{j=1}^{m-1} \left[\frac{p_{j+1}}{p_j} - 1 \right]$$

and

$$B(n) = 1 + \sum_{j=1}^{m-1}\left[1 - \frac{p_j}{p_{j+1}}\right].$$

For this purpose let

$$x_j = \frac{p_{j+1}}{p_j} - 1, \qquad j = 1, 2, 3, \cdots.$$

Then

$$\log n = \sum_{j=1}^{m-1} \log(1 + x_j).$$

From (2) and the fact that $(1/x)\log(1+x)$ is a decreasing function, we obtain

$$\limsup_{m\to\infty} \frac{\log n}{A(n)} \leqq \frac{\log(1+\theta)}{\theta}.$$

A similar argument shows that

$$\liminf_{m\to\infty} \frac{\log n}{B(n)} \geqq \frac{(1+\theta)\log(1+\theta)}{\theta}.$$

Having taken care of the above preliminaries, we turn now to the main body of the proof. Inverting the systems of equations (4a) and (4b) yields (since $R_1 = 1$)

$$\log R_n = t_n + \sum_{k=2}^{n} \frac{t_k}{k-1} \tag{5a}$$

and

$$\log R_{n+1} = u_n + \sum_{k=1}^{n-1} \frac{u_k}{k+1}. \tag{5b}$$

The values assumed by $\nu(r)$ are all terms of $\{p_m\}$; therefore

$$\log R_k = \log R_{p_{j+1}}, \qquad p_j < k < p_{j+1},$$

from which it follows that

$$t_k = \frac{p_{j+1}}{k} t_{p_{j+1}}, \qquad p_j < k \leqq p_{j+1},$$

and

$$u_k = \frac{p_j}{k} u_{p_j}, \qquad p_j \leqq k < p_{j+1}.$$

Substituting these expressions in (5a) and (5b), we obtain

$$\log R_n = t_n + \sum_{j=1}^{m-1} t_{p_{j+1}}\left[\frac{p_{j+1}}{p_j} - 1\right] \tag{6a}$$

and

$$\log R_{n+1} = u_n + \sum_{j=1}^{m-1} u_{p_j}\left[1 - \frac{p_j}{p_{j+1}}\right]. \tag{6b}$$

From (6a) and (6b) it follows (cf. [1, p. 52, Theorem 9]) that

$$\liminf_{m\to\infty} t_n \leqq \liminf_{m\to\infty} \frac{\log R_n}{A(n)} \tag{7a}$$

and

$$\limsup_{m\to\infty} u_n \geqq \limsup_{m\to\infty} \frac{\log R_{n+1}}{B(n)} \cdot \tag{7b}$$

From (7a) we have

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log \mu(r)}{\nu(r)} &\leqq \liminf_{m\to\infty} \frac{\log R_n}{A(n)} \\ &\leqq \left[\liminf_{m\to\infty} \frac{\log R_n}{\log n}\right]\left[\limsup_{m\to\infty} \frac{\log n}{A(n)}\right] \\ &\leqq \frac{\log(1+\theta)}{\rho\theta} \cdot \end{aligned}$$

Therefore

$$\liminf_{r\to\infty} [\mu(r)]^{1/\nu(r)} \leqq \alpha^{1/\rho}.$$

The remaining portion of the theorem follows similarly from (7b).

We note that α and β tend respectively to 1 and ∞ as $\theta \to \infty$. In conjunction with our theorem this remark implies the following

COROLLARY. *Suppose that* $\sum a_{p_m} z^{p_m}$ *is an entire function of positive finite order, and*

$$\lim_{m\to\infty} \frac{p_{m+1}}{p_m} = \infty.$$

Then

$$\liminf_{r\to\infty} [\mu(r)]^{1/\nu(r)} = 1$$

and

$$\limsup_{r\to\infty} [\mu(r)]^{1/\nu(r)} = \infty.$$

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MIXED BOUNDARY-VALUE PROBLEMS IN THE PLANE[1]

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Let R be a region in the plane bounded by a simple analytic curve C composed of N arcs $C_1 \cdots C_N$. Let a_m, b_m, f_m be analytic functions on C_m. Suppose $q(x, y)$ is non-negative in R. The mixed boundary-value problems discussed here require the determination of a solution of

(E) $$\Delta u - qu = 0 \quad \text{in } R,$$

(A) $$a_m u_n - b_m u = f_m \quad \text{on } C_m,$$

n the exterior normal. The problem is called regular if on each C_m either

(i) $$a_m > 0, \qquad b_m \geqq 0$$

or

(ii) $$a_m \equiv 0, \qquad b_m > 0.$$

This note presents an existence theorem based on integral equations. The method is an extension of the solution of the Dirichlet problem by simple layers as in [1] and [4]. It is intended also to provide information as to the behavior of u at the ends of the C_k.

THEOREM 1. *Every regular mixed problem has a unique solution.*

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