AN EXTENSION OF TATE'S THEOREM ON COHOMOLOGICAL TRIVIALITY[1]

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Let G be a finite group and $f\colon A \to B$ a homomorphism of G-modules. In one form, Tate's theorem says that if, for some r and all subgroups U of G, $\hat{H}^{r-1}(U, f)$ is a surjection, $\hat{H}^r(U, f)$ is an isomorphism, and $\hat{H}^{r+1}(U, f)$ is an injection, then $\hat{H}^n(U, f)$ is an isomorphism for all U and all n. Whaples has asked if the modification of this theorem stated below is true, and this paper answers Whaples' question affirmatively.

THEOREM. *If, for some integer r and every subgroup U of the finite group G, $\hat{H}^r(U, f)$ and $\hat{H}^{r+1}(U, f)$ are isomorphisms, then $\hat{H}^n(U, f)$ is an isomorphism for every n and every subgroup U.*

PROOF. By the Sylow subgroup argument in cohomology of finite groups it is sufficient to prove the theorem for p-groups. For p-groups we proceed by induction. For the trivial group the theorem is clear, so let G be a nontrivial p-group and assume the truth of the theorem for p-groups of lower order. We prove below that $\hat{H}^n(U, f)$ is an isomorphism for all U and all $n \leqq r+1$. The proof for $n \geqq r$ is analogous. By dimension shifting we may assume $r = -3$, that is, that $H_1(U, f)$ and $H_2(U, f)$ are isomorphisms for all U. (I mean the ordinary homology groups.) Let H be a maximal subgroup of G. We have the following commutative diagram with obvious vertical arrows.

$$\begin{array}{ccccccccccccc}
H_1(G/H, H_1(H, A)) & \underset{(*)}{\longrightarrow} & K_2(A) & \to & H_2(G/H, A_H) & \to & H_1(H, A)_G & \to & H_1(G, A) & \to & H_1(G/H, A_H) & \to & 0 \\
(1)\downarrow & & (2)\downarrow & & (3)\downarrow & & (4)\downarrow & & (5)\downarrow & & (6)\downarrow & & \\
H_1(G/H, H_1(H, B)) & \underset{(*)}{\longrightarrow} & K_2(B) & \to & H_2(G/H, B_H) & \to & H_1(H, B)_G & \to & H_1(G, B) & \to & H_1(G/H, B_H) & \to & 0,
\end{array}$$

where $K_2(A) = \operatorname{Coker}(i_*\colon H_2(H, A) \to H_2(G, A))$, $i\colon H \to G$ being the inclusion.

To make clear what the horizontal maps are, *and to prove the rows exact*, we make use of the homology spectral sequence

$$H_p(G/H, H_q(H, A)) \Rightarrow H_{p+q}(G, A).$$

The latter is completely dual to the usual Hochschild-Serre spectral sequence, and the edge homomorphisms $H_p(G, A) \to H_p(G/H, A_H)$

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and $H_q(H, A)_G \to H_q(G, A)$ are induced respectively by the obvious arrows $G \to G/H$ and $i: H \to G$. The exactness, then, at the last four places is just dual to the exactness of the so-called fundamental exact sequence in the cohomology of groups. The exactness at the second place and the definition of arrow (*) are derived from a slightly subtler analysis of the spectral sequence. (This remark—whose analogue holds for cohomology—was first pointed out to me by G. P. Hochschild.) Simply, if

$$0 \subset F_0 \subset F_1 \subset F_2 = H_2(G, A)$$

is the filtration associated with the spectral sequence, then $F_2/F_0 = H_2(G, A)/\mathrm{Im}\{i_*: H_2(H, A) \to H_2(G, A)\} = K_2(A)$ and $F_1/F_0 = E^{\infty}_{1,1}$. The latter, however, is a homomorphic image of $E^2_{1,1} = H_1(G/H, H_1(H, A))$ since $d^2_{1,1} = 0$.

By hypothesis the arrows (1), (4), (5) are isomorphisms. Moreover, there is the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} H_2(H, A) & \to & H_2(G, A) & \to & K_2(A) & \to & 0 \\ (7)\downarrow & & (8)\downarrow & & (2)\downarrow & & \\ H_2(H, B) & \to & H_2(G, B) & \to & K_2(B) & \to & 0. \end{array}$$

Since arrows (7), (8) are isomorphisms, so is (2). By two applications of the Five Lemma, arrows (3), (6) are isomorphisms. Thus since G/H is cyclic $H_n(G/H, f_H)$ is an isomorphism for all $n \geqq 1$.

By the induction hypothesis we may assume that $H_n(U, f)$ is an isomorphism for all proper subgroups U (in particular, for H), and for all $n \geqq 1$. Hence it suffices to show that $H_n(G, f)$ is an isomorphism for all $n \geqq 1$. To see this consider the morphism of homology spectral sequences induced by f. For the E^2 terms this gives arrows

$$H_p(G/H, H_q(H, A)) \to H_p(G/H, H_q(H, B))$$

which are isomorphisms for $(p, q) \neq (0, 0)$. This is true by the inductive hypothesis if $q > 0$, and it is what is proved above for $q = 0$. It now follows that the morphism of spectral sequences is an isomorphism, and the induced morphisms $H_n(G, f)$ $(n > 0)$ at the end of the spectral sequence are isomorphisms. This completes the proof of the theorem.

REMARKS. 1. The above theorem implies the theorem on cohomological triviality of modules. If $\hat{H}^n(U, A)$ vanishes in two successive dimensions for all subgroups U, apply the above theorem to the zero morphism of A onto 0. Since this and Tate's theorem are equivalent, we have yet another proof of Tate's theorem.

2. Let $\hat{H}^n(U, A) \cong \hat{H}^n(U, B)$ in two successive dimensions and for all subgroups but do not assume the isomorphisms induced by a module homomorphism. It would not be reasonable to expect isomorphisms for all n and all subgroups. The following counterexample justifies our pessimism. Let $G = G_p(a, b: a^3 = b^7 = 1, aba^{-1} = b^2)$; let A be Z with trivial action and B the result of dimension shifting down two steps. Then $\hat{H}^q(G, A; 7) = \hat{H}^{q-2}(G, B; 7) = 0$ for $q = 1, 2, 3, 4, 5$ and $\hat{H}^6(G, A; 7) = \hat{H}^4(G, B; 7) \neq 0$.

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QUASI-INVERTIBLE PRIME IDEALS

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In this note R will denote a commutative ring with unit and a proper ideal of R is an ideal of R different from (0) and R. Nakano has shown that R is a Dedekind domain, provided that every proper prime ideal of R is invertible [1]. In [2], Krull defines a prime ideal P to be quasi-invertible provided $PP^{-1} > P$, where $>$ denotes proper containment and P^{-1} is the set of elements x in the total quotient ring of R such that $xP \subset R$. The purpose of this note is to prove that Nakano's result remains valid when invertible is replaced by quasi-invertible. Examples are known of rank-two valuation rings in which the maximal ideal is invertible—hence, in Nakano's result, prime cannot be replaced by maximal.

LEMMA. *If P is an invertible prime ideal in R then $\bigcap_n P^n$ is a prime ideal.*

PROOF. The proof is the same as that of the first part of Theorem 4 of [1].

THEOREM. *If every proper prime ideal of R is quasi-invertible, then R is a Dedekind domain.*

PROOF. If R is a field there is nothing to prove. Let M be an arbitrary proper maximal ideal of R and denote by R_M the quotient ring of R with respect to M (see [3, pp. 218–228]). Let N denote the ideal consisting of the elements $x \in R$ such that there exists an element $m \notin M$ such that $mx = 0$. Let h be the natural homomorphism from

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