ON THE LINE GRAPH OF A PROJECTIVE PLANE[1]

A. J. HOFFMAN

1. **Introduction.** If G is a (finite, undirected) graph, its line graph (also called the interchange graph, and the adjoint graph) is the graph G^* whose vertices are the edges of G, with two vertices of G^* adjacent if the corresponding edges of G are adjacent. Let π be a projective plane with $n+1$ points on a line, and let $G(\pi)$ be the bipartite graph whose vertices are the $2(n^2+n+1)$ points and lines of π, with two vertices adjacent if and only if one of the vertices is a point, the other is a line, and the point is on the line. The graph we shall study is $(G(\pi))^*$.

For any graph G, let

$$A(G) = A = (a_{ij}) = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent vertices,} \\ 0 & \text{otherwise.} \end{cases}$$

A is called the adjacency matrix of G, and in recent years there have been several investigations to determine to what extent a regular, connected graph is determined by the characteristic roots of its adjacency matrix. In the case where G is a line graph, the following results have been obtained:

(i) If G is the line graph of the complete bipartite graph on $n+n$ vertices, and H is a regular connected graph on n^2 vertices such that $A(H)$ has the same characteristic roots as $A(G)$, then $H=G$ unless $n=4$, when there is exactly one exception [9].

(ii) If G is the line graph of the complete graph on n vertices, and H is a regular connected graph on $n(n-1)/2$ vertices, such that $A(H)$ has the same characteristic roots as $A(G)$, then $H=G$, unless $n=8$, when there are exactly three exceptions [1], [2], [3], [4], [5], [8].

In this paper, we shall prove that if H is a regular connected graph on $(n+1)(n^2+n+1)$ vertices such that $A(H)$ has the same characteristic roots as $A((G(\pi))^*)$, then $H=(G(\pi_1))^*$, where π_1 is some projective plane of the same order as π. Thus the characteristic roots of $A((G(\pi))^*)$ do determine the class of graphs $(G(\pi))^*$, but do not distinguish between projective planes of the same order.

2. **The characteristic roots of $A((G(\pi))^*)$.** It is useful first to determine the characteristic roots of $A(G(\pi))$.

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LEMMA 1. *A regular connected graph G on $2(n^2+n+1)$ vertices has, as the distinct characteristic roots of $A(G)$,*

$$(2.1) \qquad (n+1), \qquad -(n+1), \qquad \sqrt{n}, \qquad -\sqrt{n}$$

if and only if $G=G(\pi)$, where π is a projective plane of order n.

PROOF. By definition, if $G=G(\pi)$,

$$(2.2) \qquad A(G) = \begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix},$$

where B is a point-line incidence matrix of π. The characteristic roots of (2.2) are the singular values of B and their negatives. But the singular values of B are $n+1$ and $\sqrt{n}$ [7].

Conversely, assume A has (2.1) as its distinct characteristic roots. If $A=A(G)$, then [6] G is bipartite, so A is of the form (2.2), where B is a (0, 1) matrix with row and column sums equal to $n+1$, and BB^{T} has all but one characteristic root equal to n. Hence $BB^{\mathrm{T}}-nI$ is a nonnegative integral symmetric matrix of rank one with every diagonal entry equal to 1. This implies $BB^{\mathrm{T}}-nI$ has all entries 1, i.e., B is the incidence matrix of a projective plane π of order n.

Another derivation of Lemma 1 is given in the thesis of R. R. Singleton [10], in which it is proved that a regular connected graph H of valence $n+1$ and girth 6 has $2(n^2+n+1)$ vertices if and only if $H=G(\pi)$.

LEMMA 2. *The distinct characteristic roots of $A(G(\pi)^*)$ are*

$$(2.3) \qquad 2n, \qquad -2, \qquad n-1 \pm \sqrt{n}.$$

PROOF. Let $A=A((G(\pi))^*)$, B be the adjacency matrix for $G(\pi)$. Let K be the $2(n^2+n+1)$ by $(n+1)(n^2+n+1)$ matrix whose rows correspond to the points and lines of π, and whose columns correspond to the edges of $(G(\pi))^*$, i.e., each column of K contains two 1's, corresponding to an incident point and line of π, the remaining entries in the column being 0. Clearly,

$$KK^{\mathrm{T}} = (n+1)I + B, \qquad K^{\mathrm{T}}K = 2I + A.$$

The distinct characteristic roots of KK^{T} and $K^{\mathrm{T}}K$ are the same except possibly for 0. But $K^{\mathrm{T}}K$ is singular, since its rank is at most $2(n^2+n+1)$, while its order is $(n+1)(n^2+n+1)$; KK^{T} is singular, since the sum of the rows of K corresponding to points of π minus the sum of the rows of K corresponding to lines of π is the zero vector. Thus the distinct eigenvalues of KK^{T} and of $K^{\mathrm{T}}K$ are the same. Invoking (2.1) then proves (2.3).

3. THEOREM. *If G is a regular connected graph with no edges joining a vertex to itself, if G has $(n+1)(n^2+n+1)$ vertices and the adjacency matrix of G has* (2.3) *as its distinct eigenvalues, then $G=(G(\pi))^*$, for some projective plane π of order n.*

In the lemmas that follow, we assume that G satisfies the hypothesis of the theorem, $A=A(G)$, J is the matrix every entry of which is 1.

LEMMA 3. *Let*

$$(3.1)\quad P(x) = \tfrac{1}{2}(x^3 - (2n-4)x^2 + (n^2-7n+5)x + 2(n^2-3n+1));$$

then $P(A)=J$.

PROOF. It has been shown [6] that the adjacency matrix of a regular connected graph of valence d on N vertices, with distinct eigenvalues $d, \alpha_1, \cdots, \alpha_t$, satisfies $P(B)=J$, where

$$P(x) = N\prod_i (x-\alpha_i) \Big/ \prod_i (d-\alpha_i).$$

From (2.3), we then calculate (3.1).

LEMMA 4. *If two vertices of G are adjacent, then there are exactly $n-1$ vertices of G adjacent to both. If two vertices of G are not adjacent, then there are no vertices or exactly one vertex adjacent to both.*

PROOF. Let i be any vertex of G. Then i has valence $2n$, so there are $2n$ vertices $j_1, \cdots, j_{2n}$ such that $a_{ij_t}=1$, $t=1, \cdots, 2n$. We first show that

$$(3.2)\qquad \sum_t (A^2)_{ij_t} = 2n(n-1).$$

This follows from (3.1); for the left side of (3.2) is $(A^3)_{ii}$, and by (3.1), $(A^3)_{ii}=2(J)_{ii}+(2n-4)(A^2)_{ii}-(n^2-7n+5)A_{ii}-2(n^2-3n+1)$. But $J_{ii}=1$, $(A^2)_{ii}=2n$, $A_{ii}=0$, and (3.2) follows.

Next, consider the matrix

$$(3.3)\qquad B = A^2 - 2nI - (n-1)A.$$

We shall show that every entry of B is 0 or 1. Certainly every entry is an integer. Let i be any row of B. From the fact that $\sum_j (A^2)_{ij} = (2n)^2$, we infer that

$$(3.4)\qquad \sum_j b_{ij} = 2n^2.$$

We next evaluate $\sum_j b_{ij}^2 = (B^2)_{ii}$. We have from (3.3)

$$B^2 = A^4 - 2(n-1)A^3 + (n^2 - 6n + 1)A^2 + 4n(n-1)A + 4n^2 I. \tag{3.5}$$

Further, $I_{ii} = 1$, $A_{ii} = 0$, $A_{ii}^2 = 2n$, $A_{ii}^3 = 2n(n-1)$ from (3.2). To evaluate $(A^4)_{ii}$, we use (3.1), with $P(A) = J$, and obtain $AP(A) = AJ = 2nJ$. Since $AP(A)$ is a fourth degree polynomial in A, we can evaluate

$$(A^4)_{ii} = 4n - 2n(n^2 - 7n + 5) + 2n(n-1)(2n-4).$$

Putting these expressions in (3.5), we obtain

$$(B^2)_{ii} = \sum_j b_{ij}^2 = 2n^2. \tag{3.6}$$

From (3.5) and (3.6) we infer that each of the integers b_{ij} is 0 or 1. Recalling the definition of B in (3.3), this proves the second sentence of the lemma. To prove the first sentence, note from (3.2) and (3.3) that $\sum_t b_{ij_t} = 0$. Since each b_{ij} is 0 or 1, each $b_{ij_t} = 0$. By (3.3), this proves the first sentence of the lemma.

LEMMA 5. *G contains $2(n^2+n+1)$ cliques $C_1, \cdots, C_{2(n^2+n+1)}$ with the following properties:*

(3.7) *Each C_i contains exactly $n+1$ vertices.*

(3.8) *Each vertex of G is contained in exactly two C_i.*

(3.9) *Each pair of adjacent vertices of G is contained in exactly one C_i.*

PROOF. The set of cliques C_i will consist of all cliques with $n+1$ vertices, which establishes (3.7). To prove (3.9), let i and j be adjacent vertices of G. Let k and l each be adjacent to both i and j. If k and l were not adjacent, we would have a violation of the second sentence of Lemma 4. Hence, the $n-1$ vertices adjacent to both i and j (by the first sentence of Lemma 4) are adjacent to each other. These vertices, together with i and j, are the unique cliques with $n+1$ vertices containing i and j.

Let T be the total number of $n+1$ cliques, and let us count the number of incidences of cliques with pairs of vertices contained in the clique. This is

$$T\binom{n+1}{2} = \tfrac{1}{2}2n(n+1)(n^2+n+1),$$

for the right-hand side is the total number of pairs of adjacent vertices. This equation yields $T = 2(n^2+n+1)$. Thus all that remains to be proven is (3.8). Since the valence of each vertex i is $2n$, there must be at least two $n+1$ cliques containing i. If these two cliques did not contain all vertices adjacent to i, there would have to be some vertex $j\neq i$ in both cliques, violating (3.9).

We are now ready to prove the theorem. Let $\tilde{G}$ be the graph whose vertices are the $n+1$ cliques of G. Two vertices of $\tilde{G}$ are adjacent if the corresponding cliques of G have a common vertex. It follows from Lemma 5 that $\tilde{G}$ is a regular connected graph of valence $n+1$, and that $G=\tilde{G}^*$. We will be finished if we prove that $\tilde{G}=G(\pi)$. Let L be the vertex-edge incidence matrix of $\tilde{G}$, and let $\tilde{A}$ be the adjacency matrix of $\tilde{G}$. Assume $\tilde{A}$ has distinct characteristic roots $n+1$, $\alpha_1, \cdots, \alpha_t$. Since

$$LL^{\mathrm{T}} = (n+1)I + \tilde{A}, \qquad L^{\mathrm{T}}L = 2I + A,$$

and (except possibly for 0) the distinct characteristic roots of LL^{T} and $L^{\mathrm{T}}L$ are the same, it follows by the same reasoning as in Lemma 2 that the distinct characteristic roots (with the possible exception of -2) of A are

$$2n, \qquad n - 1 + \alpha_t. \tag{3.10}$$

Comparing (3.10) with (2.3), we see that, if -2 is of the form $n-1+\alpha_t$, then $\tilde{A}$ has the same distinct characteristic roots as the adjacency matrix for $G(\pi)$, and (by the "only if" part of Lemma 1) we are finished. Therefore, assume otherwise, so that (comparing (3.10) with (2.1)) we find that the distinct characteristic roots of $\tilde{A}$ are

$$n+1, \qquad \pm\sqrt{n}.$$

Since $\tilde{G}$ is regular and connected, we can, as in Lemma 3, use the theorem of [6] to assert that

$$2(\tilde{A}^2 - nI) = J.$$

But since $\tilde{A}$ is a (0, 1) matrix, this is absurd.

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INTERNATIONAL BUSINESS MACHINES CORPORATION, RESEARCH CENTER,
YORKTOWN HEIGHTS, NEW YORK