THE POLYNOMIAL OF A DIRECTED GRAPH

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1. Introduction. In a recent paper [1], the concept of the polynomial of an undirected graph was introduced, and it was pointed out that (i) a graph has a polynomial if and only if it is regular and connected, and (ii) various previous studies (see the references in [1]) were special cases of the problem: find all graphs having the same polynomial.

In this paper, we prove the analogue of (i) for directed graphs, and, in addition, obtain some results of type (ii) for a class of directed graphs arising from a mesh on a torus.

2. On the existence of polynomials. Let G be a directed graph on n vertices, with at most one edge from vertex i to vertex j, and no edge from i to i. For each vertex i, let d_i be the number of edges with terminal vertex i, e_i be the number of edges with initial vertex i. G is said to be strongly regular if $d_i = e_i = d$, $i = 1, \cdots, n$; G is said to be strongly connected if, for any vertices i and j, $i \neq j$, there is a directed path from i to j.

Let $A(G) = A$ be the adjacency matrix of G, i.e.,

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$$

Let u be the vector of order n every entry of which is unity, J the matrix of order n every column of which is u.

THEOREM 1. (i) *There exists a polynomial* $P(x)$ *such that*

$$J = P(A) \tag{2.1}$$

if and only if G is strongly connected and strongly regular.

(ii) *The unique polynomial of least degree satisfying* (2.1) *is* $nS(x)/S(d)$ *where* $(x-d)S(x)$ *is the minimal polynomial of A and d is the valence of G.*

(iii) *If $P(x)$ is that polynomial of least degree satisfying* (2.1), *then the valence of G is the greatest real root of* $P(x) = n$.

PROOF. Assume (2.1). Let i, j be distinct vertices of G. By (2.1), there is some integer k such that A^k has a positive entry in position

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(i, j), i.e., there is some k-step path from i to j. So G is strongly connected. Further, from (2.1), J commutes with A. But the (i, j)th entry of AJ is e_i, and the (i, j)th entry of JA is d_j. Thus $e_i = d_j$ for all i and j, so G is strongly regular.

To prove the converse of (i), assume G strongly connected and strongly regular. From the strong regularity, u is a left and right eigenvector of A, corresponding to the eigenvalue d. Hence, if d has multiplicity greater than 1, it must have at least one more eigenvector associated with it. But from the strong connectedness, using a standard argument [1], u is the only eigenvector corresponding to d. It follows that, if $R(x)$ is the minimal polynomial of A, and if $S(x) = R(x)/(x-d)$ then $S(d) \neq 0$. We then have

$$0 = R(A) = (A - dI)S(A). \tag{2.2}$$

Since $R(A)v = 0$ for all vectors v, it follows from (2.2) that

$$(A - dI)S(A)v = 0,$$

so $S(A)v = \alpha u$ for some α.

If $(v, u) = 0$ then $(A^k v, u) = (v, (A^T)^k u) = d^k(v, u) = 0$ for every k and so $(S(A)v, u) = 0$. Therefore, $0 = (S(A)v, u) = (\alpha u, u) = n\alpha$, i.e., $\alpha = 0$.

Thus $S(A)v = 0$ for all v such that $(v, u) = 0$; further, $S(A)u = S(d)u$. Hence $nS(A)/S(d) = J$, i.e. a polynomial which will accomplish (2.1) is

$$P(x) = \frac{n}{S(d)} S(x). \tag{2.3}$$

This completes the proof of (i); (ii) follows since (2.3) has smaller degree than the minimal polynomial of A.

To prove (iii) we note that A is non-negative and has row and column sums d. Thus, by [2], the eigenvalues of A are all of absolute value $\leq d$. The roots of $P(x)$ are eigenvalues of A and hence for real $x > d$, $|P(x)|$ is a monotone increasing function of x. From (2.3), $P(d) = n$ and so, since $P(x)$ is a real polynomial, $P(x) > n$ for $x > d$.

This completes the proof of the theorem. We call (2.3) the polynomial belonging to G (and also say that G belongs to the polynomial).

3. **A graph on a torus.** For any positive integer t let G_t be the graph whose vertices are all ordered pairs (i, j) of residues mod t and whose edges go from (i, j) to $(i, j+1)$ and $(i+1, j)$ for all i, j. Clearly G_t is strongly regular of valence 2, and strongly connected. We now derive its polynomial.

Let λ, μ be arbitrary (not necessarily distinct) tth roots of unity. Let v be the vector whose (i, j)th component is $\lambda^i\mu^j$. If A is the adjacency matrix of G_t then $Av=(\lambda+\mu)v$. Further, different vectors v_1, v_2 have as their scalar product $\sum_{i,j}\lambda_1^i\mu_1^j\lambda_2^{-i}\mu_2^{-j}$, which is zero unless $\lambda_1=\lambda_2$ and $\mu_1=\mu_2$. Hence the set of vectors corresponding to the t^2 choices of the pair λ, μ form a complete orthogonal set of right eigenvectors. From this it follows that A is normal and that the minimal polynomial of A has no repeated factors. Hence, by Theorem 1, the polynomial belonging to G_t is

$$P_t(x) = \frac{t^2}{S(d)} S_t(x), \tag{3.1}$$

where

$$S_t(x) = \frac{\prod (x-\rho)}{x-2},$$

the product being taken over all distinct ρ of the form $\lambda+\mu$, where λ, μ are tth roots of unity. For example,

$$P_2(x) = \frac{1}{2} x(x+2),$$

$$P_3(x) = \frac{1}{12} (x^3+1)(x^2+2x+4),$$

$$P_4(x) = \frac{1}{80} x(x+2)(x^2+4)(x^4+4).$$

4. **Does $P_t(x)$ characterize G_t?** In view of the investigations of comparable questions for undirected graphs, it is natural to ask: if H is a graph with t^2 vertices, and $P_t(x)$ is the polynomial of H, is $H\cong G_t$? We know of no instance in which $H\not\cong G_t$, but have only been able to prove $H\cong G_t$ if t is a prime or if $t=4$. Before specializing to those cases, however, we begin with a few lemmas. We assume H has t^2 vertices and belongs to $P_t(x)$, and A is the adjacency matrix of H.

LEMMA 1. *H is strongly connected and strongly regular of valence* 2.

PROOF. That H is strongly regular and strongly connected follows from the fact that H has a polynomial. By Theorem 1 (iii) the valence of H and the valence of G_t both equal the largest real root of $P(x)=n$ and so the valence of $H=$ the valence of $G_t=2$.

LEMMA 2. *The vertices of H can be partitioned into t sets T_i ($i\in Z_t$,* *the ring of residue classses* mod t), *such that every edge in H goes from a vertex in T_i to a vertex in T_{i+1}.*

PROOF. From the proof of Theorem 1, we know that 2 is an eigenvalue of A of multiplicity one, and every eigenvalue is of absolute value at most 2. Because 2λ is also an eigenvalue of A for λ any tth root of unity, it follows [2] that A can be conceived as having the appearance

$$\begin{pmatrix} 0 & A_0 & & & \\ & 0 & A_1 & 0 & \\ & & 0 & & \\ & & & & A_{t-2} \\ A_{t-1} & & & & 0 \end{pmatrix}, \tag{4.1}$$

where each diagonal block of 0's is square. But each A_i must also be square, since the numbers of 1's in A_i is twice the number of rows of A_i and also twice the number of columns. Thus A_i is of order t, which implies the lemma.

LEMMA 3. *Let $t>2$, let ω be a primitive tth root of unity and let λ be any tth root of unity. Then for any r, s with $(s, t)=1$, $1+\omega^r$ and $\lambda(1+\omega^{rs})$ have the same multiplicities as eigenvalues of A.*

PROOF. Let x be an eigenvector of A corresponding to the eigenvalue α, and let $x=(x_0, \cdots, x_{t-1})$ denote the partitioning of the coordinates of x corresponding to (4.1). We have $A_i x_{i+1}=\alpha x_i$. Thus $A_i(\lambda^{i+1}x_{i+1})=\alpha\lambda(\lambda^i x_i)$. Thus $(x_0, \lambda x_1, \cdots, \lambda^{t-1}x_{t-1})$ is an eigenvector of A corresponding to the eigenvalue $\lambda\alpha$. Since the minimal polynomial of A has no repeated factors, the multiplicity of an eigenvalue is just the dimension of the corresponding space of eigenvectors and so the multiplicities of α and $\lambda\alpha$ are the same. Finally the multiplicities of $1+\omega^r$ and $1+\omega^{rs}$ are the same, since these are algebraic conjugates and the characteristic polynomial of A is rational. This concludes the proof of the lemma.

Note in particular that 2λ is a simple eigenvalue of A.

LEMMA 4. *Let A be of the form* (4.1) *and of rank r. Then for $0\leqq i \leqq t-1$, $j\geqq 0$, the rank of $A_iA_{i+1}A_{i+2}\cdots A_{i+j}$ is r/t, where addition of suffixes is taken* mod t.

PROOF. Let m_i be the rank of A_i. Then $\sum_i m_i=$ the rank of $A=r$. Since the minimal polynomial of A has no repeated factors, $A=S^{-1}DS$ for some nonsingular S and diagonal D. Hence, the rank of A^t is also r. Now A^t consists of diagonal blocks $A_0A_1 \cdots A_{t-1}$, $A_1A_2 \cdots A_{t-1}A_0, \cdots, A_{t-1}A_0 \cdots A_{t-2}$. The rank of each block is at most $m = \min_i m_i$; hence, $\sum_i m_i = r = tm$ and so $m_i = r/t$ for all i. This proves the lemma for $j=0$. The result for $j>0$ follows from a similar consideration of A^j.

LEMMA 5. *Let $t>2$. If A is normal, then $H \cong G_t$.*

PROOF. Let K_i be the undirected bipartite graph whose vertices are the vertices of T_i and T_{i+1}, as defined in Lemma 2, and which has an undirected edge joining $x \in T_i$ and $y \in T_{i+1}$ if and only if an edge of H joins x to y.

We first show that K_i is a cycle of length $2t$. Since every vertex of K_i is of valence 2, K_i is the union of p_i cycles for some $p_i \geqq 1$. The matrix AA^{T} has 4 as an eigenvalue with multiplicity $\sum_{i=1}^{t} p_i$. But since A is normal, and 2λ (for λ any tth root of unity) is a simple eigenvalue of A, AA^{T} has 4 as an eigenvalue with multiplicity t. Hence $p_i = 1$, $i = 0, \cdots, t-1$, which was to be proven.

Next, we show that trace $A^t = 2t^2$. Since each K_i is a complete cycle the eigenvalues of AA^{T} are the union of the eigenvalues of t matrices of order t of the form $2I + P_i + P_i^{\mathrm{T}}$ ($i = 0, \cdots, t-1$), where each P_i is a permutation matrix that represents a single cycle on t letters. Therefore, AA^{T} has: 4 as an eigenvalue with multiplicity t; $2 + \lambda + \bar{\lambda}$ as an eigenvalue with multiplicity $2t$, for $\lambda = \exp(2\Pi ik/t)$, $k = 1, \cdots, [(t-1)/2]$; and if t is even, 0 as an eigenvalue with multiplicity t.

Since A is normal, these are the squares of the absolute values of the eigenvalues of A. Therefore, the number of eigenvalues of A of a given absolute value (other than 2 or 0) is the same for each absolute value. We also know from Lemma 3 that all eigenvalues of the same absolute value occur equally often. It follows that A has the same eigenvalues as the adjacency matrix for G_t. But the trace of the tth power of that matrix is $2t^2$, so trace $A^t = 2t^2$.

Since K_0 is a cycle of length $2t$, we may label the vertices of T_0 and T_1 as $(i, -i)$ and $(i+1, -i)$, respectively $(i \in Z_t)$, in such a way that the edges from $(i, -i)$ go to $(i+1, -i)$ and $(i, 1-i)$. Since A is normal and there is just one vertex, namely $(i, -i)$, which is the initial vertex of edges to both $(i+1, -i)$ and $(i, 1-i)$, it follows that there is just one vertex, which we label $(i+1, 1-i)$, which is the terminal vertex of edges from both $(i+1, -i)$ and $(i, 1-i)$. We now have the vertices of T_1 and T_2 labelled in such a way that the edges from $(i+1, -i)$ go to $(i+2, -i)$ and $(i+1, 1-i)$. We may continue labelling in this fashion the vertices of $T_3, T_4, \cdots, T_{t-1}$. Let p_{ij} be the number of paths of length $t-1$ from $(i, -i)$ to $(j, t-1-j)$.

Then p_{ij} is $\binom{t-1}{m}$ where m is the least positive residue (mod t) of $j-i$, for the normality of A implies that the count of paths mimics the Pascal triangle. If $(\alpha_i, t-1-\alpha_i)$ and $(\beta_i, t-1-\beta_i)$ are the vertices of T_{t-1} which are initial vertices of edges going to $(i, -i)$, then the number of paths of length t from $(i, -i)$ to itself is $p_{i,\alpha_i}+p_{i,\beta_i}$. By hypothesis, trace $A^t=2t^2$. Since the diagonal blocks of A^t are cyclic permutations of the factors $A_1, A_2, \cdots, A_t$, each block has the same trace, $2t$. Hence,

$$\sum_{i=0}^{t-1} (p_{i,\alpha_i} + p_{i,\beta_i}) = 2t.$$

Since each p_{i,α_i} and p_{i,β_i} is at least 1, it follows that p_{i,α_i} and p_{i,β_i} are exactly 1 and that α_i, β_i are just i and $i-1$. We now have that the edges from $(i, t-1-i)$ go to $(i+1, -i-1)$ and $(i, -i)$ and have completed an explicit isomorphism between G_t and H.

THEOREM 2. *If $t=2$, 4 or an odd prime, and H is a graph with t^2 vertices that belongs to $P_t(x)$, then $H\cong G_t$.*

PROOF. We shall continue to use the notations of the lemmas.

If $t=2$ the classes T_i of Lemma 2 each have 2 elements; hence the only possible distribution of edges is that of G_2.

If $t=4$ then the eigenvalues of A are ± 2, $\pm 2i$, $\pm 1 \pm i$ and 0. By Lemma 3 the eigenvalues ± 2, $\pm 2i$ are simple and the eigenvalues $\pm 1 \pm i$ have the same multiplicity, m say. Since A is of order 16 the multiplicity of 0 is $12-4m$; hence $m=1$ or 2. Suppose first that $m=1$, i.e., the multiplicity of 0 is 8. Now, by Lemma 4, each A_i is of rank 2 and must therefore be of the form P_iBQ_i, where P_i, Q_i are permutation matrices and

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Let J_4 be the 4×4 matrix of all 1's. It may be readily verified that for a permutation matrix R, BRB is one of $2B$, $2J_4-2B$ or J_4, and that J_4RB is $2J_4$. Hence

$$\begin{aligned} A_1A_2A_3A_4 &= P_1BQ_1P_2BQ_2P_3BQ_3P_4BQ_4 \\ &= 8P_1BQ_4 \quad \text{or} \quad 8P_1(J_4 - B)Q_4 \quad \text{or} \quad 4P_1J_4Q_4 = 4J_4. \end{aligned} \tag{4.2}$$

The third possibility cannot occur since, by Lemma 4, $A_1A_2A_3A_4$ is of rank 2.

Now A^4 is of the form

$$\begin{bmatrix} A_1A_2A_3A_4 & 0 & 0 & 0 \\ 0 & A_2A_3A_4A_1 & 0 & 0 \\ 0 & 0 & A_3A_4A_1A_2 & 0 \\ 0 & 0 & 0 & A_4A_1A_2A_3 \end{bmatrix}$$

and, as in the proof of Lemma 5, each of the diagonal blocks has the same eigenvalues and hence the same trace. From (4.2), the elements of $A_1A_2A_3A_4$ are divisible by 8; similarly for $A_2A_3A_4A_1$, $A_3A_4A_1A_2$ and $A_4A_1A_2A_3$. It follows therefore that the trace of A^4 is a multiple of 32. On the other hand, the trace of $A^4 = \sum \lambda^4$, the sum being over the eigenvalues λ of A. On the assumption that $m=1$ these eigenvalues are, 2, $2i$, -2, $-2i$, $1+i$, $1-i$, $-1+i$, $-1-i$, and 0 with multiplicity 8. A direct computation shows that $\mathrm{tr}(A^4)=48$. This contradicts the conclusion that $32 \mid \mathrm{tr}(A^4)$ and thus demonstrates the impossibility of the case $m=1$.

In the remaining case for $t=4$ the multiplicities of the eigenvalues are the same as those of the adjacency matrix of G_t. Hence the sum of the squares of the moduli of the eigenvalues of A is $2t^2$, which is the same as the sum of the squares of the elements of A. Therefore A is normal. By Lemma 5, $H \cong G_t$.

Finally, if t is an odd prime, the eigenvalues of A are just $2\omega^r$ and $\omega^r+\omega^s$ for $0 \leqq r < s < t$, and ω a primitive tth root of unity. We note that these numbers are all distinct. Now, by Lemma 3, $2\omega^r$ is a simple eigenvalue, and the eigenvalues $\omega^r+\omega^s$ all have the same multiplicity. This multiplicity must be 2 in order to account for all t^2 eigenvalues of A. We now have, as in the second case for $t=4$, that A is normal, and $H \cong G_t$.

References

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