ON A CLASS OF COUNTABLY PARACOMPACT SPACES

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In this note we shall characterize a topological property which is stronger than countable paracompactness but which is equivalent to it for normal spaces.

A real valued function on a topological space X is *locally bounded* if each point has a neighborhood on which the function is bounded. Let $C(X)$ denote the set of real valued continuous functions on X . A topological space is a *cb-space* if for each locally bounded function h , there exists $f \in C(X)$ such that $f \geq |h|$. J. G. Horne, Jr. initiated a study of *cb-spaces* and reported on his work in [2].

For $f \in C(X)$, the set on which f vanishes is called the *zero-set* of f and it is denoted by $Z(f)$. The complement of a zero-set is called a *cozero-set*. A *cozero cover* is a cover consisting of cozero-sets. A family of continuous functions is *locally finite* if the collection of cozero-sets associated with the family is a locally finite collection of sets. A family $\mathcal{F}$ of continuous functions is a *partition of unity* if $0 \leq f$ for all $f \in \mathcal{F}$ and $\sum_{f \in \mathcal{F}} f(x) = 1$ for all $x \in X$. A partition of unity is *subordinate* to a cover if the collection of cozero-sets associated with the partition is a refinement of the cover. A countable cover $\{U_n\}$ is an *increasing cover* if $U_n \subset U_{n+1}$ for all n . In this paper the term cover will be used to mean open cover.

THEOREM 1. *For any topological space X , the following statements are equivalent:*

- (a) *X is a cb -space.*
- (b) *Given an upper-semicontinuous function h on X , there exists $f \in C(X)$ such that $f \geq h$.*
- (c) *Given a positive (nonvanishing) lower-semicontinuous function g on X , there exists $f \in C(X)$ such that $0 < f(x) \leq g(x)$ for all $x \in X$.*
- (d) *For each countable increasing cover of X , there exists a locally finite partition of unity subordinate to that cover.*
- (e) *For each countable increasing cover of X , there exists a partition of unity subordinate to that cover.*
- (f) *Each countable increasing cover of X has a locally finite cozero refinement.*
- (g) *Each countable increasing cover of X has a σ -locally finite cozero refinement.*

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(h) *Each countable increasing cover of X has a countable cozero refinement.*

(i) *Given a decreasing sequence $\{F_n\}$ of closed sets in X with empty intersection, there exists a sequence $\{Z_n\}$ of zero-sets with empty intersection such that $Z_n \supset F_n$.*

PROOF. (a) $\rightarrow$ (b). If h is upper-semicontinuous, then h^+ is locally bounded. Thus there exists $f \in C(X)$ such that $f \geq h^+ \geq h$.

(b) $\rightarrow$ (c). If g is strictly positive and lower-semicontinuous, then g^{-1} exists and is upper-semicontinuous. Let $\phi \in C(X)$ be such that $\phi \geq g^{-1}$. Then $f = \phi^{-1} \in C(X)$ and $0 < f(x) \leq g(x)$ for all $x \in X$.

(c) $\rightarrow$ (d). For a countable increasing cover $\{U_n\}$, define $g(x) = 1$ on U_1 and $g(x) = n^{-1}$ on $U_n - U_{n-1}$ for $n > 1$. Then g is lower-semicontinuous and strictly positive. For $f \in C(X)$ such that $0 < f(x) \leq g(x)$ on X , define $\phi_n = [(n+1)f - 1]^+ [(n-1)f - 1]^-$ and $\phi = \sum \phi_n$ (here $1(x) = 1$ for all $x \in X$). Since $\{\phi_n\}$ is locally finite, $\phi \in C(X)$. Furthermore, ϕ is nonvanishing; thus ϕ is a unit of the ring $C(X)$. It now follows that $\{\phi^{-1}\phi_n\}$ is a locally finite partition of unity. That this partition is subordinate to $\{U_n\}$ is a consequence of the following: $X - Z(\phi^{-1}\phi_n) = \{x: (n+1)^{-1} < f(x) < (n-1)^{-1}\} \subset \{x: (n+1)^{-1} < g(x)\} = U_n$.

Statements (d), (e), and (f) are equivalent [6, 1.2]. Clearly (f) implies (g).

(g) $\rightarrow$ (h). Let $\mathcal{V}$ be a σ -locally finite cozero refinement of the increasing cover $\{U_n\}$. Then $\mathcal{V} = \bigcup \mathcal{V}_m$ where each $\mathcal{V}_m$ is a locally finite collection of cozero-sets. Let V_{mn} denote the union of the sets V in $\mathcal{V}_m$ such that $V \subset U_n$. Since V_{mn} is the union of a locally finite collection of cozero-sets, it is a cozero-set. Thus $\{V_{mn}\}$ is a countable cozero refinement of $\{U_n\}$.

(h) $\rightarrow$ (i). If $\{F_n\}$ is a decreasing sequence of closed sets for which $\bigcap F_n$ is empty, then $\{X - F_n\}$ is an increasing cover of X . Let A be a countable subset of $C(X)$ such that $\{X - Z(f)\}_{f \in A}$ is a refinement of $\{X - F_n\}$. Denote by Z_n the intersection of those $Z(f)$, $f \in A$ for which $Z(f) \supset F_n$. Each Z_n is a countable intersection of zero-sets and hence is a zero-set [3, §1.14]. Now $\bigcap Z_n = \bigcap_{f \in A} Z(f)$ is vacuous. Whence $\{Z_n\}$ is the desired sequence.

(i) $\rightarrow$ (a). Let h be a locally bounded function on X and define $F_n = \text{cl}\{x: |h(x)| \geq n\}$. Then $\{F_n\}$ is a decreasing sequence of closed sets. Since h is locally bounded, this sequence has empty intersection. Thus by (i) there exists a sequence $\{g_n\}$ in $C(X)$ such that $Z(g_n) \supset F_n$ and $\bigcap Z(g_n)$ is empty. Define $f_n = 1 - \bigvee_{i \leq n} n |g_i| \wedge 1$ where $1(x) = 1$ for all $x \in X$. Given $x \in X$, there exists i such that $g_i(x) \neq 0$; whence there

exists a positive integer j such that $|g_i(x)| > j^{-1}$. If $n \geq i$ and $n \geq j$, then $n|g_i(y)| \geq j|g_i(y)| > 1$ on a neighborhood of x . Thus f_n vanishes on that neighborhood provided $n \geq \max\{i, j\}$. This shows that $\{f_n\}$ is locally finite. Therefore $f = 1 + \sum f_n$ is an element of $C(X)$. On F_n , we have $g_i(x) = 0$ and $f_i(x) = 1$ for $i \leq n$. Thus $f(x) \geq n + 1 > |h(x)|$ on $F_n - F_{n+1}$. This proves that $f \geq |h|$.

REMARK 1. It should be noted that the zero-sets play the same role in the above characterization that the closed G_δ -sets play in Dowker's characterization [1] of normal and countably paracompact spaces. This is not surprising since in normal spaces the zero-sets are precisely the closed G_δ -sets.

REMARK 2. In a completely regular space, the cozero-sets form a base for the topology. Hence, in such spaces, every cover has a cozero refinement. This fact emphasizes the significance of the local finiteness and countability requirements in (f) and (h).

REMARK 3. If the word "increasing" is deleted from statements (d) through (h), and "decreasing" is deleted from (i), then each becomes a characterization of normal and countably paracompact spaces. [6, 1.1 and 1.2] or [7].

We are now able to state the following corollaries. Corollary 2 was originally proved by Horne [2].

COROLLARY 2. (i) *A cb-space is countably paracompact.*

(ii) *A normal and countably paracompact space is a cb-space.*

COROLLARY 3. *A countably compact space is a cb-space.*

COROLLARY 4. *A closed subspace of a cb-space is a cb-space.*

A subset A of a topological space X is called a *generalized cozero-set* (*generalized F_σ -set*) if each open set containing A contains a cozero-set (respectively, F_σ -set) containing A .

LEMMA 5. *In a normal space, a set is a generalized cozero-set if and only if it is a generalized F_σ -set.*

PROOF. Since a cozero-set is an F_σ -set, the "only if" part is trivial. To prove the "if" part, it suffices to show that in a normal space each F_σ -set is a generalized cozero-set. Let $A = \bigcup F_n$ where each F_n is closed. Given an open set $G \supset A$, there exists a sequence $\{Z_n\}$ of zero-sets such that $F_n \subset X - Z_n \subset G$. If $Z = \bigcap Z_n$, then Z is a zero-set and $A \subset X - Z \subset G$. Thus A is a generalized cozero-set.

THEOREM 6. *Each generalized cozero-subspace of a cb-space is a cb-space.*

PROOF. First, we shall consider a special case. Let X be a cb -space and Y be a cozero-set in X . Then there exists $g \in C(X)$, $0 \leq g$ such that $Y = X - Z(g)$. Set $F_n = \{x: n^{-1} \leq g(x)\}$. Given an upper-semicontinuous function h defined on Y , set $h_n(x) = h^+(x)$ on F_n and $h_n(x) = 0$ on $X - F_n$. Then each h_n is an upper-semicontinuous function on X ; whence there exist $f_n \in C(X)$ such that $h_n \leq f_n$. Also, for $n > 2$ there exist $g_n \in C(X)$, $g_n \geq 0$ such that $Z(g_n) \supset F_{n-2}$ and $g_n(x) = 1$ on $X - F_{n-1}$ (appropriate modifications of g will give such g_n). Set $g_1 = g_2 = 1$ and define $f = \bigvee_n (f_n g_n | Y)$. Since $\{g_n | Y\}$ is a locally finite family in $C(Y)$, f is an element of $C(Y)$. On $F_n - F_{n-1}$, $g_n(x) = 1$ and $h^+(x) = h_n(x) \leq f_n(x) \leq f(x)$. Therefore $h \leq f$. This proves the special case. The theorem follows by means of part (f) of Theorem 1, from the special case just proved.

The following previously known ([5] or [6, Corollary 1.6]) result is an immediate consequence of Lemma 5 and Theorem 6.

COROLLARY 7. *Each generalized F_σ -subspace of a normal and countably paracompact space is normal and countably paracompact.*

As noted in [2], a product of cb -spaces may fail to be cb or even countably paracompact. Also, the product of a normal and countably compact space with a compact space may fail to be normal [3, 8M.4]. The following theorem displays a property of cb -spaces which is not possessed by normal and countably paracompact spaces.

THEOREM 8. *The product of a cb -space and a locally compact, paracompact (Hausdorff) space is a cb -space.*

PROOF. First, consider the special case where Y is compact. Let $\{F_n\}$ be a decreasing sequence of closed sets in $X \times Y$ with empty intersection and let A_n be the projection of F_n into X . Since Y is compact, the projection of $X \times Y$ onto X is a closed mapping. Hence $\{A_n\}$ is a decreasing sequence of closed sets in X . Since, for each $x \in X$, the set $\{x\} \times Y$ is compact, this set fails to meet some F_n . Therefore $\{A_n\}$ has empty intersection. According to Theorem 1, there exists a sequence $\{Z_n\}$ of zero-sets in X with empty intersection such that $Z_n \supset A_n$. Then $\{Z_n \times Y\}$ is a sequence of zero-sets in $X \times Y$ with empty intersection such that $Z_n \times Y \supset F_n$. This proves the special case. Next, let Y be locally compact and paracompact. Then there exists a locally finite cover $\mathfrak{U}$ of Y consisting of relatively compact sets. Since Y is normal, there exists a mapping F from $\mathfrak{U}$ to the topology on Y such that $\text{cl} F(U) \subset U$ and $\{F(U): U \in \mathfrak{U}\}$ covers Y . Let h be a locally bounded function on $X \times Y$. By the special case just proved $X \times \text{cl} U$ is a cb -space. Thus, since $X \times F(U)$ and

$X \times (Y - U)$ are completely separated, it follows that there exists $f_U \in C(X \times Y)$ for which $f_U(x, y) \geq |h(x, y)|$ on $X \times F(U)$ while f_U vanishes outside $X \times U$. Since $\mathfrak{U}$ is locally finite, $f = \bigvee_{U \in \mathfrak{U}} f_U$ exists in $C(X \times Y)$. Clearly, $f \geq |h|$. This proves that $X \times Y$ is a *cb-space*.

We shall now show that for certain non-normal spaces the *cb-property* is equivalent to countable paracompactness. A topological space X is *pseudocompact* if every real valued continuous function on X is bounded.

THEOREM 9. *Let X be a completely regular pseudocompact space. Then the following statements are equivalent:*

- (i) X is countably compact.
- (ii) X is a *cb-space*.
- (iii) X is countably paracompact.

PROOF. The implications (i) $\rightarrow$ (ii) and (ii) $\rightarrow$ (iii) follow from Corollaries 2 and 3. We shall prove that (iii) $\rightarrow$ (i). Suppose X is countably paracompact but not countably compact. Then there is a countably infinite set $F = \{x_1, x_2, \dots, x_n, \dots\}$ which has no limit points. For each n , let U_n be an open set such that $F \cap U_n = \{x_n\}$. Then $\{U_n\} \cup \{X - F\}$ is a cover of X . This cover has a locally finite refinement $\{V_n\} \cup \{X - F\}$ such that $V_n \subset U_n$. Then $x_n \in V_n$. Let $f_n \in C(X)$ be such that f_n vanishes on $X - V_n$ while $f_n(x_n) = n$. Since $\{V_n\}$ is locally finite, $f = \bigvee f_n$ exists in $C(X)$. Clearly, f is unbounded. This is a contradiction since X is pseudocompact.

REMARK. Theorem 9 generalizes to non-normal spaces the part of Theorem 1.8 in [6] for which $m = \aleph_0$.

THEOREM 10. *Let X be a topological space. Then the following statements are equivalent.*

- (i) X is countably paracompact.
- (ii) For each increasing cover $\{U_n\}$ of X , there exists a refinement $\{V_n\}$ such that $\text{cl } V_n \subset U_n$.
- (iii) For each locally bounded function h defined on X , there exists a locally bounded lower-semicontinuous function g such that $|h| \leq g$.

PROOF. The equivalence of (i) and the closed set dual of (ii) is proved in [4].

(ii) $\rightarrow$ (iii). Let h be locally bounded and set

$$U_n = \text{int } \{x : |h(x)| \leq n\}.$$

Then $\{U_n\}$ is an increasing cover for X . Let $\{V_n\}$ be a refinement such that $\text{cl } V_n \subset U_n$ and define $g(x) = \sup \{n : x \in \bigcup_{k < n} \text{cl } V_k\}$. Then g is lower-semicontinuous and $g(x) \leq n$ on V_n . Thus g is locally

bounded. If $g(x) = n$, then $x \in \text{cl } V_n \subset U_n$; whence $|h(x)| \leq n$. Therefore $|h| \leq g$.

(iii) $\rightarrow$ (i). It suffices to show that (iii) implies that each increasing cover of X has a locally finite refinement. Let $\{U_n\}$ be an increasing cover of X and define $h(x) = \inf\{n : x \in U_n\}$. Then h is locally bounded since it is positive and upper-semicontinuous. Let g be a locally bounded, lower-semicontinuous function such that $g \geq h$. Set $V_n = \{x : n-1 < g(x)\} \cap \{x : h(x) < n+1\}$. Then $\{V_n\}$ is a refinement of $\{U_n\}$. Also, $\{V_n\}$ is locally finite since g is locally bounded.

A topological space is *extremally disconnected* if every open set has an open closure.

THEOREM 11. *Let X be an extremally disconnected space. Then X is a cb -space if and only if it is countably paracompact.*

PROOF. We need prove only the "if" part. Suppose X is countably paracompact and let $\{U_n\}$ be an increasing cover of X . Then there exists a refinement $\{V_n\}$ such that $\text{cl } V_n \subset U_n$. Since X is extremally disconnected, $\text{cl } V_n$ is open. Therefore $\{\text{cl } V_n\}$ is a cozero refinement of $\{U_n\}$. That X is a cb -space, now follows from Theorem 1.

Added in proof. A locally compact, countably paracompact space need not be a cb -space. See the example at the end of §3 in [8].

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