A SEQUENCE OF (± 1)-DETERMINANTS WITH LARGE VALUES

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1. **Introduction and statement of results.** Any positive integer m can be uniquely represented in the binary form

$$m = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_t}, \tag{1.1}$$

where $n_1 > n_2 > n_3 > \cdots n_t \geqq 0$ are integers. Put $\alpha(m) = t$, $\alpha(0) = 0$ and $A(n) = \alpha(0) + \alpha(1) + \cdots + \alpha(n)$ for $n = 0, 1, 2, \cdots$. The function $A(n)$ was studied by Cheo and Yien [1], who proved that

$$A(n) = (n/2)\log_2 n + O(n). \tag{1.2}$$

Next, let $g(n)$ be the maximum of determinants of $n \times n$ matrices with all entries $+1$ or -1. J. H. E. Cohn [2] has recently shown in a nonconstructive way that

$$n^{(n/2)(1-C(n))} \leqq g(n) \leqq n^{n/2}, \tag{1.3}$$

where $C(n)$ is non-negative and goes to 0 as $n \to \infty$.

The purpose of this paper is to gain information about $C(n)$ and the last term in (1.2). We shall prove the following theorems:

THEOREM 1. *$2^{A(n)}$ is the determinant of a square matrix of order $n+1$ with all entries $+1$ or -1, $n = 1, 2, 3, \cdots$.*

THEOREM 2. *$0 \leqq (n/2)\log_2 n - A(n-1) < (n/2)\log_2(4/3)$, $n = 1, 2, 3, \cdots$. The lower bound is attained for $n = 2^k$, $k = 0, 1, 2, \cdots$, and only then. The factor $\log_2(4/3)$ can not be replaced by a smaller number.*

The following corollary is immediate from the theorems:

COROLLARY. $n^{(n/2)(1-\log(4/3)/\log n)} < g(n) \leqq n^{n/2}$.

2. **Proof of Theorem 1.** For any two positive integers i, j let $\alpha(i, j)$ be the number of common terms in the representations (1.1) of i and j. If $i, j \geqq 0$ and at least one is 0 then put $\alpha(i, j) = 0$. Let $M(n)$ denote the matrix

$$M(n) = (a_{ij})_{i,j=0}^{n}, \qquad a_{ij} = (-1)^{\alpha(i,j)}.$$

Now we prove for $0 \leqq n < 2^k$, $k = 0, 1, 2, \cdots$,

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(2.1) $$\det M(2^k + n) = (-2)^{n+1}(\det M(2^k - 1))(\det M(n)).$$

Write $\mu' = 2^k+\mu$ and $\nu' = 2^k+\nu$, where $0 \leqq \mu,\ \nu \leqq n$. Then we get $a_{\mu'\nu} = a_{\mu\nu}$, $a_{\mu'\nu'} = -a_{\mu\nu}$. Subtracting the $(\mu+1)$st row in $M(2^k+n)$ from the $(\mu'+1)$st row, we get the row $(0, \cdots, 0, -2a_{\mu 0}, -2a_{\mu 1}, \cdots, -2a_{\mu n})$ with 2^k 0's. If we perform this subtraction for $\mu = 0, 1, \cdots, n$, the new matrix has the same determinant as $M(2^k+n)$. Now (2.1) follows by the expansion theorem of Laplace. From $\alpha(2^k+n) = \alpha(n)+1$, $0 \leqq n < 2^k$, we conclude that

(2.2) $$A(2^k + n) = A(2^k - 1) + A(n) + n + 1, \qquad 0 \leqq n < 2^k,\ k = 0, 1, \cdots.$$

By (2.1) the same relation, with "B" instead of "A," is satisfied by the function $B(n) = \log|\det M(n)|$ (logarithms are taken to the base 2). Now one easily proves by induction over k that $A(2^k+n) = B(2^k+n)$, $0 \leqq n < 2^k$, $k = 0, 1, 2, \cdots$. Thus $|\det M(n)| = 2^{A(n)}$, $n = 0, 1, 2, \cdots$. If necessary, we change signs in a row or column, and the proof is complete.

3. **Proof of Theorem 2.** The left inequality in Theorem 2 follows from Theorem 1 by the aid of Hadamard's inequality. By (2.2) we have $A(2^{k+1}-1) = 2A(2^k-1)+2^k$, $k = 0, 1, 2, \cdots$, and so, by induction,

(3.1) $$A(2^k - 1) = k2^{k-1}, \qquad k = 0, 1, 2, \cdots.$$

By (3.1), equality holds in Theorem 2 if $n = 2^k$. Since n and $A(n-1)$ are integers, equality can hold only if $n = 2^k$.

The right inequality will be proved by induction over k. We shall prove that if

(3.2) $$A(n - 1) > (n/2)\log n - (n/2)C, \qquad 1 \leqq n \leqq 2^k,$$

then this inequality holds for n' in $2^k < n' \leqq 2^{k+1}$, if C is properly chosen. Put $n' = 2^k+n$, where $1 \leqq n \leqq 2^k$. By (2.2) and (3.1) we have

(3.3) $$A(n' - 1) = k2^{k-1} + A(n - 1) + n.$$

If $f(x)$ is a convex function and $0 < a < 1$, $a+b = 1$, then $f(x+y) \leqq af(x/a)+bf(y/b)$. This yields, for the function $x \log x$ with $x = 2^k$ and $y = n$,

(3.4) $$n' \log n' \leqq k2^k + n \log n - 2^k \log a - n \log b.$$

From (3.2), (3.3) and (3.4), we then find that

$$A(n' - 1) - (n'/2)\log n' + (n'/2)C > (1 + \tfrac{1}{2}\log b)n + (C + \log a)2^{k-1}.$$

Both parentheses vanish if $b=1/4$, $a=3/4$, $C=\log(4/3)$, and (3.2) holds for n in $1 \leqq n \leqq 2^{k+1}$. The right inequality in Theorem 2 holds for $n=1$ and so for $2, 3, \cdots$.

Finally, we prove that log (4/3) cannot be replaced by a smaller number in Theorem 2. Consider the sequence

$$n_k = 4^k + 4^{k-1} + \cdots + 4^0 = (1/3)(4^{k+1} - 1), \qquad k = 0, 1, 2, \cdots.$$

By (2.2) and (3.1) we have

$$A(n_k - 1) = k4^k + A(n_{k-1} - 1) + n_{k-1}.$$

From this it follows easily by induction that

$$A(n_k - 1) = kn_k, \qquad k = 0, 1, 2, \cdots. \tag{3.5}$$

Furthermore,

$$\tfrac{1}{2} \log n_k = k + 1 - \tfrac{1}{2} \log 3 + O(4^{-k}), \qquad k = 0, 1, 2, \cdots. \tag{3.6}$$

We get, by (3.5) and (3.6),

$$A(n_k - 1) - (n_k/2)\log n_k = -(n_k/2)\log(4/3) + O(1),$$

and the theorem is proved.

References

1. P. Cheo and S. Yien, *A problem on the k-adic representation of positive integers*, Acta Math. Sinica **5** (1955), 433–438.

2. J. H. E. Cohn, *On the value of determinants*, Proc. Amer. Math. Soc. **14** (1963), 581–588.

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