

# THE COHOMOLOGY RING OF A COMPACT LIE GROUP WITH BI-INVARIANT METRIC

A. L. MAYER

**I. Introduction.** In this note we shall show that the adjoint operation  $*$  obtained from a bi-invariant riemannian metric on a compact Lie group induces an isomorphism between the cup and Pontrjagin products on the cohomology ring. This fact is easily and directly verifiable in the case of a torus, where, as we shall show elsewhere, it has interesting applications to the classical theory of abelian varieties; in fact, it motivates a definition of  $*$  on the numerical equivalence ring of an abstract polarized abelian variety.

**II. Algebraic preliminaries.** Let  $E$  be an  $n$ -dimensional vector space over  $\mathbf{R}$ ,  $\Lambda^p(E)$  the  $p$ -fold exterior product, and  $\tilde{E}$ ,  $\Lambda^p(E)^\sim$ , their respective dual spaces. There is a canonical isomorphism  $i_p: \Lambda^p(E)^\sim \rightarrow \Lambda^p(\tilde{E})$ . An orientation of  $E$  is an isomorphism  $\epsilon: \Lambda^n(E) \cong \mathbf{R}$ . It gives rise to a dual orientation  $\tilde{\epsilon}: \Lambda^n(\tilde{E}) \cong \mathbf{R}$ , and we denote the fundamental  $n$ -vector and  $n$ -covector by  $e = \epsilon^{-1}(1)$  and  $\tilde{e} = \tilde{\epsilon}^{-1}(1)$ , respectively. One defines an isomorphism  $j_p: \Lambda^p(E) \rightarrow \Lambda^{n-p}(E)^\sim$  by letting  $j_p(\alpha)(\beta) = \epsilon(\alpha \wedge \beta)$ . This gives an isomorphism  $k_p = i_{n-p} \circ j_p: \Lambda^p(E) \cong \Lambda^{n-p}(\tilde{E})$ . For  $\alpha \in \Lambda^p(E)$ ,  $\beta \in \Lambda^q(E)$ , let  $\alpha \vee \beta = k^{-1}(k\alpha \wedge k\beta) \in \Lambda^{p+q-n}(E)$ , and for  $\tilde{\alpha} \in \Lambda^p(\tilde{E})$ ,  $\tilde{\beta} \in \Lambda^q(\tilde{E})$ ,  $\tilde{\alpha} \bar{\wedge} \tilde{\beta} = \alpha \vee k^{-1}(\tilde{\beta}) \in \Lambda^{p+q-n}(E)$ .

The composition map  $T: E \oplus E \rightarrow E$ , sending  $\alpha \oplus \beta$  into  $\alpha + \beta$ , can be uniquely extended to an algebra homomorphism  $\{T^p\}$ ,  $T^p: \Lambda^p(E \oplus E) \rightarrow \Lambda^p(E)$ . Since  $\Lambda^p(E \oplus E) \cong \bigoplus_{r+s=p} \Lambda^r(E) \otimes \Lambda^s(E)$ , we can define, for  $\alpha \in \Lambda^r(E)$  and  $\beta \in \Lambda^s(E)$ ,  $T^{r+s}(\alpha \otimes \beta) \in \Lambda^{r+s}(E)$ , and a simple computation shows this to equal  $\alpha \wedge \beta$ . Also,  $T: E \oplus E \rightarrow E$  may be dualized to give  $\tilde{T}: \tilde{E} \rightarrow \tilde{E} \oplus \tilde{E}$ , which extends uniquely to an algebra homomorphism  $\{\tilde{T}^p\}$ ,  $\tilde{T}^p: \Lambda^p(\tilde{E}) \rightarrow \Lambda^p(\tilde{E} \oplus \tilde{E})$ . Let  $\tilde{e}' = T^n(\tilde{e}) \in \Lambda^n(\tilde{E} \oplus \tilde{E})$ . Given  $\alpha \in \Lambda^p(E)$ ,  $\beta \in \Lambda^q(E)$ , we have  $\alpha \otimes \beta \in \Lambda^{p+q}(E \oplus E)$  and  $\alpha \otimes \beta \bar{\wedge} \tilde{e}' \in \Lambda^{p+q-n}(E \oplus E)$ . An easy computation shows that  $T^{p+q-n}(\alpha \otimes \beta \bar{\wedge} \tilde{e}') = \alpha \vee \beta$ .

A quadratic form on  $E$  is an isomorphism  $\phi: E \rightarrow \tilde{E}$ , extendable uniquely to an algebra isomorphism  $\{\phi^p\}$ ,  $\phi^p: \Lambda^p(E) \rightarrow \Lambda^p(\tilde{E})$ . We define  $*$ :  $\Lambda^p(E) \rightarrow \Lambda^{n-p}(E)$  by  $*\alpha = k^{-1}\phi\alpha$ . Note that  $*(\alpha \wedge \beta) = k^{-1}(\phi\alpha \wedge \phi\beta) = k^{-1}(kk^{-1}\phi\alpha \wedge kk^{-1}\phi\beta) = k^{-1}(k*\alpha \wedge k*\beta) = *\alpha \vee *\beta$ . So  $*$  is an isomorphism of the  $\wedge$ -algebra onto the  $\vee$ -algebra; if  $\phi$  is unitary, i.e.,  $*1 = e$ , the map is unit preserving.

---

Received by the editors February 2, 1963.

Needless to say, the above considerations extend to the case where  $E$  is a vector bundle over a manifold  $M$ ; e.g., if  $E$  is the cotangent bundle of an oriented manifold, we recover the familiar  $*$  of Weitzenback-Hodge.

**III. Integration along the fibre.** Let  $G$  be a compact oriented Lie group with dual Lie algebra  $E$  (whose orientation is  $\epsilon$ ), and let  $\Phi: Y^k \rightarrow X^m$  be a principal fibre bundle with group  $G$ ,  $X$  and  $Y$  being compact and having orientations compatible with that of  $G$  (with respect to the local product structure). The orientation of  $E$  gives  $G$  a unique left-invariant Haar measure. For all  $x \in X$ , there is a homeomorphism  $\psi_x: G \rightarrow \Phi^{-1}(x)$ , sending the invariant  $n$ -vector  $\bar{\epsilon}$  onto an invariant  $n$ -vector  $\bar{\epsilon}_0$  along the fibre. Let  $\alpha$  be a  $p$ -form on  $Y$ ; we define a  $(p-n)$ -form  $*\Phi(\alpha)$  on  $X$  by integrating along the fibre. Precisely,  $\alpha \wedge \bar{\epsilon}_0$ , along the fibre  $\Phi^{-1}(x)$  is a form annihilating all vectors tangent to the fibre; hence it may be integrated, by integrating its coefficients with respect to the Haar measure, to give a  $(p-n)$ -form  $*\Phi(\alpha)_x$  at  $x$ . The local product structure and the invariance of the measure guarantee that  $*\Phi(\alpha)$  is well defined and differentiable, and it is easy to check that  $*\Phi(\alpha)$  is closed if  $\alpha$  is.

Let  $y$  be a  $p$ -cohomology class in  $Y$ , the de Rham class of a form  $\alpha$ . Its Poincaré dual class  $Py \in H_{k-p}(Y, \mathbf{R})$  is defined as a linear functional on  $H^{k-p}(Y, \mathbf{R})$  by  $Py(z) = \int_Y \alpha \wedge \beta$ ,  $z$  being the de Rham class of a form  $\beta$ . The map  $\Phi$  induces a map  $\Phi_*: H_{k-p}(Y, \mathbf{R}) \rightarrow H_{k-p}(X, \mathbf{R})$  by letting  $\Phi_*(Py)(w) = \int_Y \alpha \wedge \Phi^* \gamma$ , where  $w$  is the de Rham class of a  $(k-p)$ -form  $\gamma$  on  $X$ . On the other hand, the Poincaré dual class of  $*\Phi(\alpha)$  is defined as a functional on  $H^{k-p}(X, \mathbf{R})$  by sending the de Rham class  $w$  of a form  $\gamma$  into  $\int_X *\Phi(\alpha) \wedge \gamma$ . A partition of unity and Fubini's theorem show immediately that  $P[*\Phi(\alpha)] = \Phi_*(Py)$ ,  $y$  again being the class of  $\alpha$ . Hence  $*\Phi$  induces on the de Rham cohomology the well-known Gysin homomorphism.

**IV. The convolution product.** Now let  $X=G$ ,  $Y=G \times G$  and  $\Phi$  be the composition map. Further, let  $\pi_1$  and  $\pi_2$  be the projections of  $G \times G$  onto its factors. If  $\alpha$  is a  $p$ -form, and  $\beta$  a  $q$ -form, on  $G$ , the  $(p+q-n)$ -form  $\alpha \bullet \beta = *\Phi(\pi_1^* \alpha \wedge \pi_2^* \beta)$  will be called the convolution of  $\alpha$  and  $\beta$ . (If  $p=q=n$ ,  $\alpha = fe$  and  $\beta = ge$ , then  $\alpha \bullet \beta = he$ , and it is clear that the function  $h$  is the convolution, in the usual sense, of the functions  $f$  and  $g$ .) From the last remarks in §III, it is clear that the convolution algebra on closed forms induces on the cohomology of  $G$  the algebra structure obtained by transposing (via Poincaré duality) the Pontrjagin algebra on homology to cohomology.

Now let  $\alpha$  be right invariant, and  $\beta$  left invariant. Then  $\pi_1^* \alpha$

$\wedge \pi \beta_2^* \bar{\lambda} \bar{e}_0$  is invariant along the fibre  $\Phi^{-1}(1)$  so the value of  $\alpha \bullet \beta$  at 1 is merely the image under the canonical map  $\wedge^{p+q-n}(E \oplus E) \rightarrow \wedge^{p+q-n}(E)$  of the value of  $\pi_1^* \alpha \wedge \pi_2^* \beta \bar{e} \bar{\lambda}_0$  at  $1 \times 1$ . (We are using the fact that the map  $\psi_1: G \rightarrow G \times G$  induces, on the dual Lie algebra, the canonical map of composition.) By the remarks in II,  $\alpha \bullet \beta$  evaluated at 1 is thus (the value of  $\alpha$  at 1)  $\vee$  (the value of  $\beta$  at 1).

Now assume that both  $\alpha$  and  $\beta$  are bi-invariant. Then  $\pi_1^* \alpha \wedge \pi_2^* \beta$  is bi-invariant on  $G \times G$ ; invariant, in particular, under the liftings via  $\Phi$  of both right and left translations on  $G$ . Hence,  $\alpha \bullet \beta$  is bi-invariant, and so equals  $\alpha \vee \beta$ .

**V. Conclusion of the proof.** We may map  $H^*(G, \mathbf{R})$  into  $\wedge^*(E)$  by sending each class into the unique bi-invariant form it contains, evaluated at 1. It is well known that this is a homomorphism of the cup-algebra into the  $\wedge$ -algebra, and we have shown above that it is also a homomorphism of the (transposed) Pontrjagin algebra into the  $\vee$ -algebra.

Now, let  $\phi$  be a bi-invariant riemannian metric on  $G$ . (One is easily constructed by integrating a positive-definite quadratic form on  $\bar{E}$  under the adjoint group, and then translating.) Then (e.g., since bi-invariant forms are harmonic) the action of  $*$  on  $\wedge^*(E)$  is compatible with the action of  $*$  on cohomology. Letting  $\oplus$  denote the (cohomological) Pontrjagin product, we may combine this last remark with that at the end of §II to assert that, for any cohomology classes  $x$  and  $y$  on  $G$ ,  $*(x \cup y) = (*x) \oplus (*y)$ .

COLUMBIA UNIVERSITY