NOTE ON THE UNITS OF A REAL QUADRATIC FIELD

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Let p denote a prime $\equiv 1 \pmod{4}$. It is known that, if $\theta = e^{2\pi i/p}$,

$$\frac{\prod_b (1 - \theta^b)}{\prod_a (1 - \theta^a)} = \eta$$

(here b and a run over the quadratic nonresidues and quadratic residues, respectively, that lie between 0 and p) is a unit of the real quadratic field $R(\sqrt{p})$, and that $\eta > 1$.

The fact that $\eta > 1$ is usually deduced from the theory of the class-number of quadratic fields. We present a short proof independent of the theory of the class-number. As in the paper of Chowla and Mordell [*Note on the nonvanishing of $L(1)$* , Proc. Amer. Math. Soc. 12 (1961), 283–284], we have $\eta \neq 1$ since $L(1) \neq 0$ (proved in the paper cited) in the relation [ibid.]

$$(1) \quad \eta = \exp\{SL(1)\},$$

where $L(1) = \sum_1^\infty (n|p)n^{-1}$ and $S = \sum_1^p (n|p)\theta^n$. Here $(n|p)$ is Legendre's symbol.

From the theory of the Gaussian sum,

$$(2) \quad S = \sqrt{p}.$$

Also,

$$L(1) = \lim_{s \rightarrow 1+0} L(s) \geq 0$$

since

$$L(s) = \sum_1^\infty (n|p)n^{-s} > 0 \quad (s > 1)$$

from the Euler product for $L(s)$. It now follows from Chowla and Mordell that

$$(3) \quad L(1) > 0.$$

From (1), (2), (3) we obtain $\eta > 1$. Q.E.D.

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