ON THE UNDECIDABILITY OF POWER SERIES FIELDS

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Let F be a field and $F((T))$ the field of formal power series over F.

THEOREM. *If F is undecidable, then $F((T))$ is undecidable.*

REMARK. Malcev [**1**] obtained special cases of this result.

PROOF. It suffices to show that the valuation subring $A=F[[T]]$ is elementarily definable in $F((T))$ [**3**, Sec. 6]. Fix $m>1$ such that $\mathrm{char}(F)\nmid m$. An idea in J. Robinson [**3**] shows that A is definable in terms of T: $A=$the set of x such that $\exists y[y^m=1+Tx^m]$. By compounding this idea and another trick we can get rid of T: $A=$the set of x such that $\exists w\exists y\forall u\forall x_1\forall x_2\exists z\forall y_1\forall y_2[(z^m=1+wx_1^m x_2^m\vee y_1^m\neq 1+wx_1^m\vee y_2^m\neq 1+wx_2^m)\wedge u^m\neq w\wedge y^m=1+wx^m]$. Indeed, this follows from the fact that $A=\bigcup_{w\in G}A_w$, where $A_w=$the set of x such that $\exists y[y^m=1+wx^m]$, if $G\subset F((T))$ has the following properties: (1) $T\in G$; (2) for each $w\in G$, A_w is closed under multiplication and its elements have poles of bounded order. (1) shows that $\bigcup_{w\in G}A_w\supset A$ while (2) gives the reverse inclusion.

COROLLARY. *If* $\mathrm{char}(F)=0$, *then F is decidable if and only if $F((T))$ is decidable.*

PROOF. Combine the above theorem with Theorem 6 of Ax-Kochen [**4**].

Finally, we note that the theorem, the corollary and their proofs remain valid if $F((T))$ is replaced by any Hensel field valued in a Z-group with residue class field F.

REFERENCES

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