ON POWERS OF REAL NUMBERS (mod 1)

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Suppose λ and θ are real numbers, $\lambda > 0$ and $\theta > 1$. Denote by d_n the unique integer satisfying $\lambda\theta^n < d_n + 1/2 \leqq \lambda\theta^n + 1$, so that d_n is the "nearest" integer to $\lambda\theta^n$, and put $\epsilon_n = \lambda\theta^n - d_n$. It is not known for which pairs λ, θ does $\epsilon_n \to 0$ as $n \to \infty$. If $\sum_{n=0}^{\infty} |\epsilon_n|^2 < \infty$, or if θ is algebraic and $\epsilon_n \to 0$, then θ is a PV number [2] (i.e. an algebraic integer whose conjugates have absolute value less than one) and λ is a number in the field of θ. It follows from [1] and the Fatou-Hurwitz theorem, that if the function $\sum_{n=0}^{\infty} \epsilon_n z^n$ is of bounded characteristic in $|z| < 1$, then θ is an algebraic integer, and that all of its conjugates have absolute value no greater than one. Finally, by a result of Pisot, if $\lambda > 1$ and $|\epsilon_n| \leqq [2e\theta(\theta+1)(1+\log \lambda)]^{-1}$ for all $n \geqq 0$, then θ is an algebraic integer [3].

The purpose of this note is to give a slight generalization of the last-mentioned result of Pisot and obtain new arithmetic conditions on the $|\epsilon_n|$ which will guarantee that θ is algebraic.

Put $\delta_{nk} = \sup_{m \geq k} \sum_{i=m}^{m+n} |\epsilon_i|$ and $\delta_n = \delta_{n,0}$.

THEOREM. *Suppose that for some $n \geqq 1$,*

$$\lambda^{1/n}\delta_n < \frac{\theta - 1}{4(\theta + 1)\theta^2} \tag{1}$$

and

$$\lambda\theta^n \geqq \delta_n(\theta + 1). \tag{2}$$

Then θ is an algebraic integer whose conjugates have absolute value no greater than 1.

PROOF. By (1),

$$\frac{1}{2\lambda^{1/n}\delta_n(\theta + 1)} > \frac{2^{1/n}\theta^2}{\theta - 1},$$

hence by (2),

$$\frac{1}{\lambda^{1/n}}\left(\frac{1}{\delta_n(\theta + 1)} - \frac{1}{2\lambda\theta^n}\right) > \frac{2^{1/n}\theta^2}{\theta - 1}$$

or

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$$\left(\frac{1}{2\lambda}\right)^{1/n}\left(\frac{\theta-1}{\theta}\right)\left(\frac{1}{\delta_n(\theta+1)}-\frac{1}{2\lambda\theta^n}\right)>\theta. \tag{3}$$

Now let a and b be positive real numbers satisfying $a<1/(2\lambda)$, $b<((\theta-1)/\theta)(1/\delta_n(\theta+1)-1/2\lambda\theta^n)$,

$$a^{1/n}b>\theta. \tag{4}$$

They exist by (3). Denote by $x=(x_0, x_1, \cdots, x_n)$ a point in Euclidean $(n+1)$-dimensional space. The volume of the region satisfying

$$\left|\sum_{i=0}^{n} x_i\theta^i\right|<a \tag{5}$$

and

$$|x_i|<b, \qquad \text{for } \theta \leqq i \leqq n-1, \tag{6}$$

is $2^{n+1}b^na/\theta^n>2^{n+1}$ by (4). Since the region defined by (5) and (6) is convex and symmetric with respect to 0, there exists, by Minkowski's convex-body theorem, a point x with integral coordinates satisfying (5) and (6). By (5) and (6),

$$\begin{aligned}|x_n| &< \theta^{-n}(a+x_0+x_1\theta+\cdots+x_{n-1}\theta^{n-1})\\ &< \frac{a}{\theta^n}+\frac{b}{\theta-1}\cdot\end{aligned}$$

Hence $|x_i|<a/\theta^n+b\theta/(\theta-1)$ for $0\leqq i\leqq n$. Now,

$$\begin{aligned}\left|\sum_{i=0}^{n} x_id_i\right| &\leqq \lambda\left|\sum_{i=0}^{n} x_i\theta^i\right|+\left|\sum_{i=0}^{n} x_i\epsilon_i\right|\\ &< \lambda a+\left(\frac{a}{\theta^n}+\frac{b\theta}{\theta-1}\right)\delta_n.\end{aligned}$$

Since $\lambda a<1/2$, and

$$\begin{aligned}\left(\frac{a}{\theta^n}+\frac{b\theta}{\theta-1}\right)\delta_n &< \left(\frac{1}{2\lambda\theta^n}+\left(\frac{1}{\delta_n(\theta+1)}-\frac{1}{2\lambda\theta^n}\right)\right)\delta_n\\ &< \frac{1}{\theta+1}\\ &< \frac{1}{2},\end{aligned}$$

$|\sum_{i=0}^{n} x_id_i|<1$ and, being an integer, is 0. We now proceed inductively. Suppose $\sum_{i=0}^{n} x_i d_{m-1+i}=0$. Since $d_r=\lambda\theta^r-\epsilon_r=\theta(\lambda\theta^{r-1}-\epsilon_{r-1})-\epsilon_r+\theta\epsilon_{r-1}=\theta d_{r-1}-\epsilon_r+\theta\epsilon_{r-1}$, we have

$$\begin{aligned}\left|\sum_{i=0}^{n} x_i d_{m+i}\right| &\leqq \theta\left|\sum_{i=0}^{n} x_i d_{m-1+i}\right| \\ &\quad + \left|\sum_{i=0}^{n} x_i \epsilon_{m+i}\right| + \theta\left|\sum_{i=0}^{n} x_i \epsilon_{m-1+i}\right| \\ &< (1+\theta)\left(\frac{a}{\theta^n}+\frac{b\theta}{\theta-1}\right)\delta_n \\ &< 1.\end{aligned}$$

Since $\sum_{i=0}^{n} x_i d_{m+i}$ is an integer, it is 0. It follows that $\sum_{m=0}^{\infty} d_m z^m$ is a rational function; by the Fatou-Hurwitz theorem, it can be written in the form $A(z)/B(z)$ where $A(z)$ and $B(z)$ are polynomials with integral coefficients and $B(0)=1$. Since $1/\theta$ is a root of $B(z)$, θ is an algebraic integer; since $A(z)/B(z)-\lambda/(1-\theta z)=\sum_{m=0}^{\infty}\epsilon_m z^m$ is regular in the unit disk, $1/\theta$ is the only root of $B(z)$ in the unit disk; hence all conjugates of θ have absolute value no greater than one.

REMARK. If $\lambda \geqq 1$, then condition (1) of the theorem implies condition (2).

COROLLARY. *Suppose* $\lim_n \sup \sum_{m=n}^{2n} |\epsilon_m| < (\theta-1)/4(\theta+1)\theta^3$. *Then θ is an algebraic integer whose conjugates have absolute value no greater than one.*

PROOF. Clearly $\delta_{n,n}<(\theta-1)/4(\theta+1)\theta^3$, for all sufficiently large n. Put $\lambda'=\lambda\theta^n$ and apply the theorem to λ' and θ. Condition (1) becomes $\lambda^{1/n}\theta\delta_{n,n}<(\theta-1)/4(\theta+1)\theta^2$, while condition (2) is satisfied whenever n is large enough. Since $\lambda^{1/n}\to 1$ as $n\to\infty$, the hypotheses of the theorem are satisfied.

REFERENCES

1. D. Cantor, *Power series with integral coefficients*, Bull. Amer. Math. Soc. **69** (1963), 362–366.

2. J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge Univ. Press, New York, 1957.

3. C. Pisot, *Repartition* (mod 1) *des puissances successives des nombres réels*, Comment. Math. Helv. **19** (1946–1947), 153–160.

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