

# ON MEASURE AND OTHER PROPERTIES OF A HAMEL BASIS

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F. B. Jones has established [2] several interesting measure-related properties of a Hamel basis. In particular, he established the existence of a Hamel basis which contains a nonempty perfect set (and, hence, supports a nontrivial Borel measure) and commented on the plausibility of the notion that a Hamel basis should be in some sense "thick." Our purpose is to complement Jones' results by showing, subject to the continuum hypothesis which we assume throughout, that there exists a Hamel basis which intersects each first category set in, at most, a countable set and, hence, has universal measure zero (cf. [1]). It then follows, for instance, that the sum  $E + F = \{e + f; e \in E, f \in F\}$  of a universal null set  $E$  and a universal null set  $F$  need not be a universal null set and, moreover, an iterate  $E^n$  of  $E$  need not be Lebesgue measurable (cf. [2]).

In order to fulfill our purpose it suffices to establish the following theorem. (We wish to acknowledge collaboration with R. E. Zink on problems related to the content of this note.)

**THEOREM.** *There exists a Hamel basis  $H$  which intersects each perfect nowhere dense set in, at most, a countable set.*

Before proceeding to a proof of the theorem we wish to state the following lemma which we shall have occasion to use.

**LEMMA.** *If  $Q$  is a first category subset of  $(0, 1]$  and  $x$  is a point of  $(0, 1)$ , then there exists a point  $y$  of  $(0, x)$  such that  $x + y \in (x, 1)$  and neither  $y$  nor  $x + y$  is an element of  $Q$ .*

**PROOF OF THEOREM.** Let  $\Omega$  denote the first uncountable ordinal and let  $\{P_\alpha\}_{\alpha < \Omega}$  and  $\{x_\alpha\}_{\alpha < \Omega}$ ,  $x_1 = 1$ , be well orderings of the perfect nowhere dense subsets of  $(0, 1]$  and the points of  $(0, 1]$ . We shall define  $H = \bigcup_{\alpha < \Omega} H_\alpha$  inductively as follows. Let  $H_1 = \{1\}$ ,  $R_1 = \emptyset$ . Suppose  $1 < \alpha < \Omega$  and, for  $1 \leq \beta < \alpha$ ,  $H_\beta$  and  $R_\beta$  satisfy:

- (1)  $H_\beta$  is, at most, countable.
- (2) The elements of  $H_\beta$  are linearly independent over the rationals.
- (3)  $R_\beta$  is a subset of the linear span  $H_\beta^L$  of  $H_\beta$ .
- (4)  $H_\beta \cap R_\beta = \emptyset$ .

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- (5)  $(H_\beta \cup R_\beta) \supset \bigcup_{\gamma \leq \beta} \{x_\gamma\}$ .
- (6)  $R_\beta \subset \bigcup_{\gamma \leq \beta} \{x_\gamma\}$ .
- (7)  $H_\gamma \subset H_\beta$ ,  $R_\gamma \subset R_\beta$ ,  $\gamma < \beta$ .
- (8) If  $x \in H_\beta - H_\gamma$ , then  $x \notin P_\gamma$  ( $\beta > \gamma$ ).

In order to simplify what follows, let  $K_\alpha = \bigcup_{\beta < \alpha} H_\beta$ ,  $S_\alpha = \bigcup_{\beta < \alpha} R_\beta$ , and  $T_\alpha = \bigcup_{\beta < \alpha} P_\beta$ . We now consider cases:

- (a) If  $x_\alpha \in K_\alpha$ , let  $H_\alpha = K_\alpha$  and  $R_\alpha = S_\alpha$ .
- (b) If  $x_\alpha \in K_\alpha^L - K_\alpha$ , let  $H_\alpha = K_\alpha$  and  $R_\alpha = S_\alpha \cup \{x_\alpha\}$ .
- (c) If  $x_\alpha \notin K_\alpha^L \cup T_\alpha$ , let  $H_\alpha = K_\alpha \cup \{x_\alpha\}$  and  $R_\alpha = S_\alpha$ .
- (d) If  $x_\alpha \in T_\alpha - K_\alpha^L$  and there exists a rational number  $r$  such that  $rx_\alpha \in (0, 1) - T_\alpha$ , let  $\lambda$  be the least index such that  $x_\lambda$  is a rational multiple of  $x_\alpha$  in  $(0, 1) - T_\alpha$  and then let  $H_\alpha = K_\alpha \cup \{x_\lambda\}$  and  $R_\alpha = S_\alpha \cup \{x_\alpha\}$ .
- (e) If  $\{x_\alpha\}^L \cap (0, 1) \subset T_\alpha - K_\alpha^L$ , let  $Q = (K_\alpha \cup \{x_\alpha\})^L \cup T_\alpha$  and apply the lemma to obtain the least index  $\lambda$  such that  $x_\alpha$  and  $x_\lambda$  play the role of  $x$  and  $y$  of the lemma. Then let  $H_\alpha = K_\alpha \cup \{x_\lambda\} \cup \{x_\lambda + x_\alpha\}$  and  $R_\alpha = T_\alpha \cup \{x_\alpha\}$ .

#### BIBLIOGRAPHY

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