A CLASS OF RIEMANN SURFACES[1]

THOMAS A. ATCHISON

In this paper, a class of open simply connected Riemann surfaces is considered and the uniformizing function and its derivative are exhibited in an infinite product representation. An infinite product of the form of the uniformizing function is then shown to produce a surface of this class.

Definition of the class of surfaces. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_{3n-2}\}_{n=1}^{\infty}$, and $\{b_{3n-1}\}_{n=1}^{\infty}$ be three sequences of real numbers such that for every positive integer n, $a_n>0$, $b_{3n-2}>0$, and $b_{3n-1}>0$; $0<a_{3n-2}<b_{3n-2}$; and $0<a_{3n-1}<b_{3n-1}$. For each sheet, a copy of the Riemann sphere, let a surface F consist of sheets $S_1, S_2, \cdots$, over the Riemann sphere such that

(1) S_1 is slit from a_1 to b_1,

(2) for n odd, S_{3n-1} is slit from $-b_{3n-1}$ to $-a_{3n-1}$ and from a_{3n-2} to b_{3n-2}; for n even, S_{3n-1} is slit from $-b_{3n-2}$ to $-a_{3n-2}$ and from a_{3n-1} to b_{3n-1},

(3) for n odd, S_{3n} is slit from $-b_{3n-1}$ to $-a_{3n-1}$ and from a_{3n} to $+\infty$; for n even, S_{3n} is slit from $-\infty$ to $-a_{3n}$ and from a_{3n-1} to b_{3n-1}, and

(4) for n odd, S_{3n+1} is slit from $-b_{3n+1}$ to $-a_{3n+1}$ and from a_{3n} to $+\infty$; for n even, S_{3n+1} is slit from $-\infty$ to $-a_{3n}$ and from a_{3n+1} to b_{3n+1}.

S_n is joined to S_{n+1} by connecting the slits which have one endpoint at $\pm a_n$ to form first-order branch points at the endpoints of the slits.

The uniformizing function. F is simply connected and open, hence by the General Uniformization Theorem, there exists a unique function ϕ such that ϕ maps F one-one and conformally onto $\{z \mid |z|<R \leqq \infty\}$, where for $w=f(z)=\phi^{-1}(z)$, $f(0)=0\in S_1$, and $f'(0)=1$.

Let α_i denote the zeros of $f'(z)$ corresponding to the first order branch points over $(-1)^{i+1}a_i$, while $-\beta_{3i-2}$ and $-\beta_{3i-1}$ denote the zeros of $f'(z)$ corresponding to the first-order branch points over $(-1)^{i+1}b_{3i-2}$ and $(-1)^i b_{3i-1}$, respectively. Let $f(\delta_i)=0\in S_i$ for $i=2, 3, \cdots$, let $f(\gamma_{3i})=\infty$, a first-order branch point over ∞ on S_{3i} and S_{3i+1}, let $f(\gamma_1)=\infty\in S_1$, and let $f(\gamma_{3i-1})=\infty\in S_{3i-1}$.

Received by the editors November 16, 1963 and, in revised form, April 13, 1964.

[1] This is a condensation of the author's doctoral dissertation written at The University of Texas under the direction of Professor H. B. Curtis, Jr.

LEMMA 1. *f is real for real z, and for* $k \geqq 1$,

$$0 < \alpha_k < \delta_{k+1} < \alpha_{k+1}$$

and

$$0 > -\gamma_1 > -\beta_{3k-2} > -\gamma_{3k-1} > -\beta_{3k-1} > -\gamma_{3k} > -\beta_{3k+1}.$$

PROOF. The argument is essentially the same as that of [3, pp. 511–512].

LEMMA 2. *For each positive integer* n, *let* F_n *be the first* $3n+3$ *sheets of* F *with the slit from* $(-1)^n a_{3n+3}$ *to* $(-1)^n \infty$ *deleted. Then there exists a rational function which maps the z-plane onto* F_n.

PROOF. F_n is a simply connected closed surface with branch points over $a_1, -a_2, \cdots, (-1)^{n+1}a_{3n+2}, b_1, -b_2, \cdots, (-1)^{n+1}b_{3n+2}$ and with n branch points over ∞. F_n has $3n+3$ points over the origin and $n+3$ points over ∞ which are not branch points. Then F_n is the Riemann surface of the inverse of a unique rational function, $w = R_n(z)$, such that $R_n(0) = 0 \in S_1$, $R_n'(0) = 1$, and $R_n(\infty) = \infty \in S_{3n+3}$. If $R_n(\delta_{k,n}) = 0 \in S_k$ for $2 \leqq k \leqq 3n+3$; $R_n(-\gamma_{1,n}) = \infty \in S_1$, $R_n(-\gamma_{3k-1,n}) = \infty \in S_{3k-1}$, $R_n(-\gamma_{3k,n}) = \infty \in S_{3k}$, $R_n(-\gamma_{3n+2,n}) = \infty \in S_{3n+2}$ for $1 \leqq k \leqq n$; $R_n'(\alpha_{k,n}) = 0$ for $1 \leqq k \leqq 3n+2$; $R_n'(-\beta_{3k-2,n}) = 0$ for $1 \leqq k \leqq n+1$; and $R_n'(-\beta_{3k-1,n}) = 0$ for $1 \leqq k \leqq n+1$, then

$$R_n(z) = \frac{z}{\gamma^*_{1,n}} \prod_{k=1}^{n} \frac{(\delta^*_{3k-1,n})(\delta^*_{3k,n})(\delta^*_{3k+1,n})}{(\gamma^*_{3k-1,n})(\gamma^*_{3k,n})^2} \frac{(\delta^*_{3n+2,n})(\delta^*_{3n+3,n})}{(\gamma^*_{3n+2,n})}$$

and

$$R_n'(z) = \frac{1}{(\gamma^*_{1,n})^2} \frac{\prod_{k=1}^{3n+2} (\alpha^*_{k,n}) \prod_{k=1}^{n+1} (\beta^*_{3k-2,n}) \prod_{k=1}^{n+1} (\beta^*_{3k-1,n})}{\prod_{k=1}^{n+1} (\gamma^*_{3k-1,n})^2 \prod_{k=1}^{n} (\gamma^*_{3k,n})^3}$$

where $\delta^*_{j,n} = 1 - z/\delta_{j,n}$, $\gamma^*_{j,n} = 1 + z/\gamma_{j,n}$, $\alpha^*_{j,n} = 1 - z/\alpha_{j,n}$ and $\beta^*_{j,n} = 1 + z/\beta_{j,n}$.

LEMMA 3. *F is parabolic.*

PROOF. Let D_n be the z-plane slit along the real axis from $\alpha_{3n+2,n}$ to $+\infty$. Then D_n is mapped by $w = R_n(z)$ onto F_n with the sheet S_{3n+3} slit from $(-1)^{n+1}a_{3n+2}$ to $(-1)^n \infty$ along the real axis. But $\zeta = \phi(w)$ maps this cut surface one-to-one on the domain Δ_n of the ζ-plane bounded by the curve C_{3n+3} and the segment $(\alpha_{3n+2}, \alpha_{3n+3})$ and containing $\zeta=0$. Thus $\zeta=\phi[R_n(z)]=\psi_n(z)$ provides a schlicht map of D_n onto Δ_n with $\psi_n(0)=0$ and $\psi_n'(0)=1$. As in the argument of [6, p. 55], the distance from $\zeta=0$ to the curve C_{3n+3} is greater than $\alpha_{3n+2,n}$.

For $0<z<\alpha_{1,n}$,

$$\frac{1}{\overset{*}{\gamma}_{1,n}} \frac{\prod_{k=1}^{n+1} \overset{*}{\beta}_{3k-2,n}}{\prod_{k=1}^{n} \overset{*}{\gamma}_{3k,n}} < 1, \quad \prod_{k=1}^{n+1} \overset{*}{\gamma}_{3k-1,n} > 1, \quad \prod_{k=1}^{n} (\overset{*}{\gamma}_{3k,n})^2 > 1,$$

and

$$\prod_{k=1}^{3n+2} \overset{*}{\alpha}_{k,n} > 0.$$

Thus, if

$$\frac{1}{\bar{\alpha}_{3n+2}} = \frac{1}{3n+2} \sum_{k=1}^{3n+2} \frac{1}{\alpha_{k,n}},$$

then

$$0 < R_n'(z) <, \prod_{k=1}^{3n+2} \overset{*}{\alpha}_{k,n} \leqq \left[\frac{1}{3n+2} \sum_{k=1}^{3n+2} \overset{*}{\alpha}_{k,n}\right]^{3n+2} = (1 - z/\bar{\alpha}_{3n+2})^{3n+2}.$$

Hence,

$$a_1 = \int_0^{\alpha_{1,n}} R_n'(z)dz < \int_0^{\alpha_{1,n}} (1 - z/\bar{\alpha}_{3n+2})^{3n+2} dz$$
$$< \int_0^{\bar{\alpha}_{3n+2}} (1 - z/\bar{\alpha}_{3n+2})^{3n+2} dz = \frac{\bar{\alpha}_{3n+2}}{3n+3}.$$

But

$$\sum_{k=1}^{3n+2} \frac{1}{\alpha_{k,n}} = \frac{3n+2}{\bar{\alpha}_{3n+2}} < \frac{1}{a_1},$$

thus for $1 \leqq i \leqq 3n+2$,

$$\frac{i}{\alpha_{i,n}} < \sum_{k=1}^{3n+2} \frac{1}{\alpha_{k,n}} < \frac{1}{a_1},$$

or $ia_1 < \alpha_{i,n}$ for $1 \leqq i \leqq 3n+2$, $n=1, 2, \cdots$. Therefore, the distance from the origin to C_{3n+3} is greater than $(3n+2)a_1$ for all n, and F is parabolic.

LEMMA 4. *$R_n(z)\to f(z)$ uniformly on compact subsets of the plane as $n\to\infty$.*

LEMMA 5. *For all $k\geqq 1$, $\alpha_{k,n}\to\alpha_k$, $\beta_{3k-2,n}\to\beta_{3k-2}$, $\beta_{3k-1,n}\to\beta_{3k-1}$, $\gamma_{3k-1,n}\to\gamma_{3k-1}$, $\gamma_{3k,n}\to\gamma_{3k}$, and $\delta_{k,n}\to\delta_k$ as $n\to\infty$.*

PROOF. These lemmas are proved in essentially the same way as similar results are obtained in [3].

LEMMA 6. $\limsup_{j\to\infty}\sum_{k=1}^{j} 1/d_{k,n}<\infty$ *and* $\sum_{k=1}^{\infty} 1/d_k<\infty$ *for the following cases*: $d_{k,n}=\alpha_{k,n}$ *with* $j=3n+2$; $d_{k,n}=\beta_{3k-2,n}$ *with* $j=n+1$; $d_{k,n}=\beta_{3k-1,n}$ *with* $j=n+1$; $d_{k,n}=\gamma_{3k-1,n}$ *with* $j=n+1$; $d_{k,n}=\gamma_{3k,n}$ *with* $j=n$; $d_k=\alpha_k$; $d_k=\beta_{3k-2}$; $d_k=\beta_{3k-1}$; $d_k=\gamma_{3k-1}$; and $d_k=\gamma_{3k}$. *Also* $\limsup_{n\to\infty}\sum_{k=2}^{3n+3} 1/\delta_{k,n}<\infty$ *and* $\sum_{k=2}^{\infty} 1/\delta_k<\infty$.

PROOF. If C_n denotes the coefficient of z in the Taylor expansion of $\log R_n'(z)$ about the origin, then $C_n\to K<\infty$ as $n\to\infty$ and thus, because $0<\gamma_{1,n}<\beta_{1,n}$, $0<\gamma_{3k-1,n}<\beta_{3k-1,n}$, and $0<\gamma_{3k,n}<\beta_{3k+1,n}$,

$$-\infty<C_n<-\sum_{k=1}^{3n+2} 1/\alpha_{k,n}-\sum_{k=1}^{n+1} 1/\beta_{3k-1,n}-1/\beta_{1,n}-\sum_{k=2}^{n+1} 2/\beta_{3k-2,n}<0.$$

Consequently, the first three cases are established. The remaining cases follow from the inequalities

$$0<\beta_{3k-2,n}<\gamma_{3k-1,n},\quad 0<\beta_{3k-1,n}<\gamma_{3k,n},$$

and

$$0<\alpha_{k,n}<\delta_{k+1,n}.$$

LEMMA 7. *If*

$$\pi(z)=\frac{1}{(\gamma_1^*)^2}\,\frac{\prod_{k=1}^{\infty}\alpha_k^*\prod_{k=1}^{\infty}\beta_{3k-2}^*\prod_{k=1}^{\infty}\beta_{3k-1}^*}{\prod_{k=1}^{\infty}(\gamma_{3k-1}^*)^2\prod_{k=1}^{\infty}(\gamma_{3k}^*)^3}$$

then $f'(z)=\exp(\delta'z)\pi(z)$ *where* $\delta'=\lim s_n'$ *with* s_n' *the coefficient of z in the Taylor expansion of* $\log R_n'(z)/\pi(z)$ *about the origin.*

PROOF. Using the ordering of α, β, and γ and Lemmas 4 and 5, $\log R_n'(z)/\pi(z)\to\delta'z$ as $n\to\infty$.

LEMMA 8. $\delta'=0$.

PROOF. The inequality $\delta'\leqq 0$ may be demonstrated using methods similar to those of [2]. Because the factors of $\pi(z)$ are canonical products of genus zero, then for every $\epsilon>0$ and for $0<\rho\leqq|\arg z|\leqq\pi-\rho$, $\pi(z)=O(e^{\epsilon|z|})$ and $1/\pi(z)=O(e^{\epsilon|z|})$. Thus under the same conditions, for R sufficiently large and $|z|>R$, $\exp(\delta' \operatorname{Re}(z)-\epsilon|z|)\leqq|f'(z)|\leqq\exp(\delta' \operatorname{Re}(z)+\epsilon|z|)$. For $\delta'<0$ and $|z|>R$, there exists $\phi>0$ such that for $-5\pi/6\leqq\arg z\leqq-2\pi/3$, $|f'(z)|\geqq\exp(\phi|z|)$ and for $-\pi/3\leqq\arg z\leqq-\pi/6$, $\exp(-\phi|z|)\geqq|f'(z)|$.

Since the distance from the origin to $C_{n+1}\to\infty$ as $n\to\infty$, then there exists $\{r_n\}_{n=1}^{\infty}$ such that $r_n\to\infty$ as $n\to\infty$ and for every z on C_{n+1}, $|z|\geqq r_n$. Let $z_{1,2j}$ and $z_{2,2j}$ be two points on C_{2j} such that $\arg z_{1,2j}=-5\pi/6$ and $\arg z_{2,2j}=-2\pi/3$. As z traverses C_{2j} from $z_{1,2j}$ to $z_{2,2j}$, f is real and increasing and hence $f'(z)dz\geqq0$. If $\delta'<0$, then for ζ_1 and ζ_2 in $\{\zeta|\ -\pi/3\leqq\arg\zeta\leqq-\pi/6,\ |\zeta|>R\}$,

$$|f(\zeta_2)-f(\zeta_1)|=\left|\int_{\zeta_1}^{\zeta_2}f'(t)dt\right|\leqq\int_{\zeta_1}^{\zeta_2}|f'(t)|\ |dt|$$
$$\leqq\exp(-\phi R)\,|\zeta_2-\zeta_1|\,.$$

Therefore $f(z)\to K$, a constant, uniformly in $\{z|\ -\pi/3\leqq\arg z\leqq-\pi/6,\ |z|>R\}$ as $z\to\infty$. As $z\to\infty$ along the ray $\arg z=-\pi/4$, $f(z)<0$ when the ray crosses C_{2n} and $f(z)>0$ when the ray crosses C_{2n+1}. Hence $K=0$, and for j sufficiently large, $0>f(z_{1,2j})>f(z_{4,2j})>-1$, where $\arg z_{4,2j}=-\pi/4$ and $z_{4,2j}$ is on C_{2j}. For r_{2j} sufficiently large,

$$b_{2j}-a_{2j}\geqq f(z_{2,2j})-f(z_{1,2j})=\int_{z_{1,2j}}^{z_{2,2j}}f'(t)\,dt\geqq\exp(\phi\,r_{2j})\pi/6\,r_{2j}.$$

Thus as $j\to\infty$, $f(z_{2,2j})-f(z_{1,2j})\to\infty$. But $f(z_{2,2j})\leqq0$, and hence $f(z_{1,2j})\to-\infty$, which contradicts $0>f(z_{1,2j})>-1$ for j sufficiently large.

LEMMA 9. *If*

$$P(z)=\frac{z}{\gamma_1^*}\,\frac{\prod_{k=2}^{\infty}\delta_k^*}{\prod_{k=1}^{\infty}(\gamma_{3k-1})\prod_{k=1}^{\infty}(\gamma_{3k}^*)^2},$$

then $f(z)=\exp(\delta z)P(z)$, *where* δ *is real and* $\delta=\lim_{n\to\infty}s_n$ *with* s_n *the coefficient of* z *in the Taylor expansion of* $\log R_n(z)/P(z)$ *about the origin.*

PROOF. Log $[R_n(z)/P(z)]\to\delta z$ uniformly on any compact subset of the plane as $n\to\infty$.

LEMMA 10. $\delta=0$.

PROOF. Using Lemma 4, as $n \to \infty$,

$$
\begin{aligned}
0 \leqq \limsup_{n\to\infty} \left| s_n - s_n' \right| &\leqq \limsup_{n\to\infty} \left| \sum_{k=m}^{n+1} 1/\gamma_{3k-1,n} - \sum_{k=m}^{n+1} 1/\beta_{3k-2,n} \right| \\
&+ \limsup_{n\to\infty} \left| \sum_{k=m}^{\infty} (1/\gamma_{3k-1} - 1/\beta_{3k-2}) \right| \\
&+ \limsup_{n\to\infty} \left| \sum_{k=m}^{n} 1/\gamma_{3k,n} - \sum_{k=m}^{n+1} 1/\beta_{3k-1,n} \right| \\
&+ \limsup_{n\to\infty} \left| \sum_{k=m}^{\infty} (1/\gamma_{3k} - 1/\beta_{3k-1}) \right| \\
&+ \limsup_{n\to\infty} \left| \sum_{k=m}^{3n+3} 1/\delta_{k,n} - \sum_{k=m}^{3n+2} 1/\alpha_{k,n} \right| \\
&+ \limsup_{n\to\infty} \left| \sum_{k=m}^{\infty} (1/\delta_k - 1/\alpha_k) \right| \leqq \limsup_{n\to\infty} 1/\gamma_{3m-1,n} \\
&+ 1/\gamma_{3m-1} + \limsup_{n\to\infty} 1/\beta_{3m-1,n} + 1/\beta_{3m-1} + \limsup_{n\to\infty} 1/\delta_{m,n} \\
&+ 1/\delta_m = 2/\gamma_{3m-1} + 2/\beta_{3m-1} + 2/\delta_m, \quad \text{for every } m \geqq 2.
\end{aligned}
$$

Therefore, $\delta = \delta' = 0$.

Collecting the above results, we have the following theorem.

THEOREM. *A surface of the above class is parabolic and the mapping function is given by* $f(z) = P(z)$ *where* $f'(z) = \pi(z)$. *Also* $\sum_{k=2}^{\infty} 1/d_k$ *converges for* $d_k = \alpha_k$, $d_k = \beta_{3k-2}$, $d_k = \beta_{3k-1}$, $d_k = \gamma_{3k-1}$, $d_k = \gamma_{3k}$, *and* $d_k = \delta_k$.

The remainder of the paper proves the following theorem.

THEOREM. *Let*

$$
f(z) = \frac{z}{\gamma_1^*} \prod_{k=1}^{\infty} \frac{(\overset{*}{\delta}_{3k-1})(\overset{*}{\delta}_{3k})(\overset{*}{\delta}_{3k+1})}{(\gamma_{3k-1}^*)(\gamma_{3k}^*)^2},
$$

where $\sum_{k=2}^{\infty} 1/\delta_k$ *and* $\sum_{k=1}^{\infty} 1/\gamma_k$ *converge, and for every integer* k, $0 < \delta_{k+1} < \delta_{k+2}$ *and* $0 < \gamma_1 < \gamma_{3k-1} < \gamma_{3k} < \gamma_{3k+2}$. *The Riemann surface of the inverse of* $f(z)$ *is of the class described above.*

LEMMA 11. *There exists a sequence of rational functions* $R_n(z)$ *such that* $R_n(z) \to f(z)$ *uniformly on compact subsets of the plane and such that the paths other than the real axis on which* $R_n(z)$ *is real are* $3n$ *simple, closed, nonintersecting curves each symmetric with respect to the real axis.*

PROOF. Consider

$$R_n(z) = \frac{z}{\gamma_1^*}\prod_{k=1}^{n}\frac{(\overset{*}{\delta}_{3k-1})(\overset{*}{\delta}_{3k})(\overset{*}{\delta}_{3k+1})}{(\gamma_{3k-1}^*)(\gamma_{3k}^*)^2}\,.$$

Using Rolle's theorem, at least $3n$ zeros of $R_n'(z)$ are determined such that $0<\alpha_{1,n}<\delta_2<\alpha_{2,n}< \cdots <\alpha_{3n,n}<\delta_{3n+1}$, at least $2n$ zeros of $R_n'(z)$ are determined such that $0>-\gamma_1>-\beta_{1,n}>-\gamma_2>-\beta_{2,n}> \cdots >-\beta_{2n+2,n}>-\gamma_{3n}$, and $R_n(z)$ has n first-order branch points over the poles at $-\gamma_{3k}$. The indicated critical points of $R_n(z)$ account for the total branch order, $6n$, of the rational function.

Through each value of $\alpha_{k,n}$, $\beta_{k,n}$, and $\gamma_{3k,n}$ passes a curve in addition to the real axis on which $R_n(z)$ is real. Since $R_n(\bar{z}) = [R_n(z)]^-$, where $[\ ]^-$ means complex conjugate, these curves are symmetric about the real axis, and because ∞ is not a critical point, the curves are simple, closed, nonintersecting ones each of which intersects the real axis at two points. A consideration of the order of $\alpha_{k,n}$, $\beta_{k,n}$, and $\gamma_{3k,n}$ will show that the $3n$ curves, $C_{k,n}$, on which $R_n(z)$ is real intersect the real axis at $\alpha_{k,n}$ and $-\beta_{k,n}$ or $\alpha_{3k,n}$ and $-\gamma_{3k,n}$.

LEMMA 12. *Any ray from the origin intersects each curve $C_{k,n}$ exactly once.*

PROOF. $R_n(z) = P_n(z)/Q_n(z)$ where $\deg P_n(z) = 3n+1$ and $\deg Q_n(z) = 3n+1$. The condition that z is a point on $C_{k,n}$ or the real axis is that

$$2i\mathfrak{J}[R_n(z)] = R_n(z) - [R_n(z)]^- = P_n(z)/Q_n(z) - P_n(\bar{z})/Q_n(\bar{z}) = 0.$$

Hence on $C_{k,n}$, $F(x, y) = P_n(z)Q_n(\bar{z}) - P_n(\bar{z})Q_n(z) = 0$. $F(x, y)$ is of degree at most $6n+1$ in x and y simultaneously. Any line $y = mx$ or $x = my$ intersects each $C_{k,n}$ at least twice, and these $6n$ intersections together with one at the origin make a total of $6n+1$ intersections, the maximum number of solutions of $F(x, mx) = 0$ and $F(my, y) = 0$.

LEMMA 13. *The points of $C_{k,n}$ tend to the points of a curve C_k as $n\to\infty$ where C_k intersects the real axis at α_k and $-\beta_k$ or $-\gamma_k$. Any ray from the origin intersects C_k exactly once, C_k is symmetric about the real axis, and C_k does not pass through $z=\infty$.*

PROOF. This lemma is demonstrated in a manner similar to the demonstration of Lemma 14 of [2].

LEMMA 14. *$f(z)$ is a schlicht and conformal map of the upper half of the annular region between C_j and C_{j+1} onto $\mathfrak{J}\{(-1)^j w\} > 0$.*

PROOF. This follows using Darboux's theorem.

References

1. L. V. Ahlfors and Leo Sario, *Riemann surfaces*, Princeton Univ. Press, Princeton, N. J., 1960.
2. H. B. Curtis, Jr., *Some properties of functions which uniformize a class of simply connected Riemann surfaces*, Proc. Amer. Math. Soc. 10 (1959), 525–530.
3. ———, *The uniformization of a class of simply connected Riemann surfaces*, Proc. Amer. Math. Soc. 11 (1960), 511–516.
4. G. R. MacLane, *Riemann surfaces and asymptotic values associated with real entire functions*, The Rice Institute Pamphlet, Special Issue, November, 1952.
5. Rolf Nevanlinna, *Eindeutige analytische Funktionen*, Springer, Berlin, 1936.
6. H. E. Taylor, *Determination of the type and properties of the mapping function of a class of doubly-connected Riemann surfaces*, Proc. Amer. Math. Soc. 4 (1953), 52–68.

University of Texas and
Texas Technological College

THE ADJOINT OF A DIFFERENTIAL OPERATOR WITH INTEGRAL BOUNDARY CONDITIONS

ALLAN M. KRALL

In [1] a second-order differential operator was defined on those functions in $L^2(0, \infty)$ satisfying an integral-point type of boundary condition. An analysis of its spectrum and two "eigenfunction" expansions follows. Left unanswered was the problem of finding the adjoint operator and explaining where the nonhomogeneous expansion came from. We now derive the adjoint operator, classify its spectrum and show that the nonhomogeneous expansion is, in fact, the eigenfunction expansion associated with the adjoint operator. It is interesting to see that the adjoint operator is a combination of a differential operator and a one-dimensional vector in $L^2(0, \infty)$.

1. The operator L. We consider a differential expression of the form $ly = -y'' + q(x)y$, $0 \leqq x < \infty$, where $q(x)$ is an arbitrary measurable complex function satisfying $\int_0^\infty |q(x)|\, dx < \infty$.

We denote by D_0 those functions f defined on $[0, \infty)$ and satisfying

1. f is in $L^2(0, \infty)$,
2. f' exists and is absolutely continuous on every finite subinterval of $[0, \infty)$,
3. lf is in $L^2(0, \infty)$.

Let $K(x)$ be an arbitrary complex-valued function on $L^2(0, \infty)$,

Received by the editors April 15, 1964.