PERTURBATION OF A STURM-LIOUVILLE OPERATOR BY A FINITE FUNCTION

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If T_1 is a self-adjoint operator and V is a bounded self-adjoint operator in a Hilbert space and if $T_2 = T_1 + V$, then Theorem 1 of [2] states that

$$\lim_{\tau\to\infty} \tau^2[S\{R_2(i\tau) - R_1(i\tau)\} + S\{R_1(i\tau)VR_1(i\tau)\}] = 0 \tag{1}$$

provided

$$\|\,|V|^{1/2}R_1(i\tau)\|_2 = O(\tau^{-\alpha}) \quad \text{as} \quad \tau\to\infty, \qquad \text{where } \alpha > 1/2. \tag{2}$$

Here $R_i(z)$ is the resolvent of T_i, $\|\ \|_2$ is the Schmidt norm, and S stands for trace. From (1) various trace formulas for differential operators may be obtained. In [2] the condition (2) was verified for the situation in which T_1 is defined in $L^2[0, \infty)$ by the ordinary differential operator $L = -D^2$ and the boundary condition $u(0) = 0$, and V is the operator of multiplication by $p(x)$, where p is real, continuous, bounded, and absolutely integrable on $[0, \infty)$. Recently M. G. Gasymov [1] derived trace formulas for the case that $L = -D^2 + q$, where $q(x)\to\infty$ as $x\to\infty$, and $p(x)$ is a finite[1] function. Gasymov's article suggests that condition (2) is valid for the case considered by him. It is the purpose of this article to show that if p is a finite function and if T is bounded below, then (2) holds in fact, whatever the behavior of q at infinity is. The method employed is similar to that used by B. M. Levitan [3] for the study of the spectral function of L.

THEOREM. *Let q be a real-valued continuous function on $[0, \infty)$. Let T be a self-adjoint operator in $L^2[0, \infty)$ defined by $L = -D^2 + q$ and the boundary condition $u'(0) = 0$. (If L is in the limit circle case at infinity, a boundary condition at infinity is also included.) Let $p(x)$ be a real-valued continuous function on $[0, \infty)$ which vanishes for $x > A$. If T is bounded below, $\|\,|V|^{1/2}R(i\tau)\|_2 = O(\tau^{-3/4})$ as $\tau\to\infty$, where V is the operator of multiplication by p, the norm is the Schmidt norm and $R(z)$ is the resolvent of T.*

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[1] By a *finite* function is meant a function which vanishes outside some bounded set.

PROOF. If $\psi(x,\lambda)$ is the solution of $Lu=\lambda u$, $u(0)=1$, $u'(0)=0$, and if $\rho(\lambda)$, $-\infty<\lambda<\infty$, is the spectral function of T, it is shown in [2] that

$$\| |V|^{1/2}R(i\tau)\|_2^2 = \int_0^\infty |p(x)| \int_{-\infty}^\infty \frac{|\psi(x,\lambda)|^2}{\lambda^2+\tau^2}\, d\rho(\lambda)dx. \tag{3}$$

Assuming that T is bounded below by 0 and using the finiteness of p, (3) can be written

$$\| |V|^{1/2}R(i\tau)\|_2^2 = \int_0^A |p(x)| \int_0^\infty \frac{|\psi(x,\lambda)|^2}{\lambda^2+\tau^2}\, d\rho(\lambda)dx. \tag{4}$$

If q is extended evenly to the negative real numbers, then $\psi(x,\lambda)$ is extended evenly to the negative real numbers also, and it may be verified that $\psi(x,\lambda)\cos\sqrt{\lambda}t$ is the solution of an initial value problem for a hyperbolic equation over the whole (x,t)-plane. $\psi(x,\lambda)\cos\sqrt{\lambda}t$ may therefore be written in the form

$$\begin{aligned}\psi(x,\lambda)\cos\sqrt{\lambda}t &= (1/2)[\psi(x+t,\lambda)+\psi(x-t,\lambda)]\\ &\quad + (1/2)\int_{x-t}^{x+t}\psi(s,\lambda)w(x,t,s)\,ds,\end{aligned} \tag{5}$$

where $w(x,t,s)$ is constructed from the Riemann function of the hyperbolic equation and is continuous in (x,t,s) for $-\infty<x,t,s<\infty$.

Let g be a real-valued function defined on $[0,1]$ having the following properties: $g\in C^2$, $g(0)=1$, $g(1)=g'(0)=g'(1)=0$. If we multiply both sides of (5) by $e^{-at}g(t)$ and integrate from 0 to 1, we obtain, after some changes of variables, a reversal of order of integration, and use of the equation $\psi(x,\lambda)=\psi(-x,\lambda)$,

$$\psi(x,\lambda)\int_0^1 \cos\sqrt{\lambda}t\, e^{-at}g(t)\,dt = \int_0^{x+1} F(x,s,a)\psi(s,\lambda)\,ds, \quad x\geqq 0, \tag{6}$$

where

$$\begin{aligned}F(x,s,a) = (1/2)\Big[&e^{-a|s-x|}g(|s-x|)+e^{-a|s+x|}g(|s+x|)\\ &+\int_{|x-s|}^1 w(x,t,s)e^{-at}g(t)\,dt\\ &+\int_{|x+s|}^1 w(x,t,-s)e^{-at}g(t)\,dt\Big],\end{aligned}$$

$$0\leqq x\leqq 1,\ 0\leqq s\leqq 1-x;$$

$$F(x, s, a) = (1/2)\left[e^{-a|s-x|}g(|s-x|) + \int_{|x-s|}^{1} w(x, t, s)e^{-at}g(t)\,dt\right],$$
$$0 \leqq x \leqq 1,\ 1 - x \leqq s \leqq 1 + x;$$
$$= 0, \qquad 1 \leqq x,\ 0 \leqq s \leqq x - 1;$$
$$= (1/2)\left[e^{-a|s-x|}g(|s-x|) + \int_{|x-s|}^{1} w(x, t, s)e^{-at}g(t)\,dt\right],$$
$$1 \leqq x,\ x - 1 \leqq s \leqq x + 1.$$

Since the map $\hat{f}(\lambda) = \int_0^\infty \psi(x, \lambda)f(x)\,dx$ is an isometry of $L^2[0, \infty)$ onto $L^2(\rho)$, it follows from (6) that

$$\int_0^\infty \left|\psi(x,\lambda) \int_0^1 \cos\sqrt{\lambda}t\, e^{-at}g(t)\,dt\right|^2 d\rho(\lambda) = \int_0^{x+1} |F(x, s, a)|^2\,ds. \tag{7}$$

Two integrations by parts show that

$$\int_0^1 g(t)\cos\sqrt{\lambda}t\, e^{-at}\,dt = \frac{a}{a^2+\lambda}\left[1 + \int_0^1 g'(t)e^{-at}\cos\sqrt{\lambda}t\,dt - \frac{\sqrt{\lambda}}{a^2+\lambda}\int_0^1 g''(t)e^{-at}\sin\sqrt{\lambda}t\,dt - \frac{\lambda}{a(a^2+\lambda)}\int_0^1 g''(t)e^{-at}\cos\sqrt{\lambda}t\,dt\right].$$

Hence, there is a number $a_0 > 0$ such that

$$\int_0^1 g(t)\cos\sqrt{\lambda}t\, e^{-at}\,dt \geqq \frac{a}{2(a^2+\lambda)} \quad \text{for } \lambda \geqq 0 \text{ and } a \geqq a_0,$$

and therefore

$$\left|\int_0^1 g(t)\cos\sqrt{\lambda}t\, e^{-at}\,dt\right|^2 \geqq \frac{a^2}{4(a^2+\lambda)^2} \geqq \frac{a^2}{8(a^4+\lambda^2)} \text{ for } \lambda \geqq 0,\ a \geqq a_0.$$

It follows from (7) that

$$\frac{a^2}{8}\int_0^\infty \frac{|\psi(x,\lambda)|^2}{a^4+\lambda^2}\,d\rho(\lambda) \leqq \int_0^{x+1} |F(x, s, a)|^2\,ds, \quad a \geqq a_0. \tag{8}$$

Multiplying (8) by $|p(x)|$, integrating from 0 to A, and assuming $A > 1$, one obtains

$$\frac{a^2}{8}\int_0^A |p(x)|\int_0^\infty \frac{|\psi(x,\lambda)|^2}{a^4+\lambda^2}\,d\rho(\lambda)dx$$

$$(9)\quad \leqq \int_0^1 |p(x)|\left\{\int_0^{1-x}|F(x,s,a)|^2\,ds+\int_{1-x}^{1+x}|F(x,s,a)|^2\,ds\right\}dx$$

$$+\int_1^A |p(x)|\int_{x-1}^{x+1}|F(x,s,a)|^2\,dsdx,\quad a\geqq a_0.$$

By use of the triangle inequality and the continuity of the functions involved, one may show that the right side of (9) is $O(1/a)$ as $a\to\infty$. Letting $a=\tau^{1/2}$, it follows from (4) and (9) that $\||V|^{1/2}R(i\tau)\|_2 = O(\tau^{-3/4})$. This proves the theorem in the case that T is bounded below by zero.

Now suppose that $T\geqq -m$, $m>0$. Let $T_1=T+mE$, where E is the identity. Then T_1 is bounded below by zero. Also, T_1 is an ordinary differential operator of the same type as T. It follows from what we have already shown that $\||V|^{1/2}R_1(i\tau)\|_2=O(\tau^{-3/4})$. Since

$$\begin{aligned} R(i\tau) = R_1(i\tau)[E+mR(i\tau)],\ \|\,|V|^{1/2}R(i\tau)\|_2 &\leqq \|\,|V|^{1/2}R_1(i\tau)\|_2\|E+mR(i\tau)\| \\ &\leqq \|\,|V|^{1/2}R_1(i\tau)\|_2(1+m/\tau). \end{aligned}$$

Thus the theorem is completely proved.

REMARK. The situation in which T is defined by more general boundary conditions or L has singularities at both ends of the interval can be handled similarly.

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