A MOMENT PROBLEM IN L_1 APPROXIMATION

CHARLES R. HOBBY[1] AND JOHN R. RICE[2]

1. **Introduction.** The purpose of this paper is to show the existence of a solution to a certain moment problem which arises in the study of approximation in the L_1 norm

$$\|f\| = \int_0^1 |f| \, d\mu.$$

Let n be a fixed, but arbitrary, integer and consider the sign function $s(A, x) = s(A)$ defined on $[0, 1]$ by

$$(1) \qquad s(A, x) = \begin{cases} +1 & x \in (a_i, a_{i+1}) \quad i \text{ even}, \\ 0 & x = a_i, \\ -1 & x \in (a_i, a_{i+1}) \quad i \text{ odd} \end{cases}$$

where A stands for the vector $(a_1, a_2, \cdots, a_n)$ and the convention is made that $a_i \leqq a_{i+1}$, $a_0 = 0$, $a_{n+1} = 1$. Thus $s(A)$ is simply a step function taking on the values ± 1 with at most n sign changes. Let μ be a finite, nonatomic measure on $[0, 1]$ such that every $s(A)$ is measurable and let $\{\phi_i | i = 1, 2, \cdots, n\}$ be n functions integrable on $[0, 1]$. We may now state the

MOMENT PROBLEM. *Determine A^* so that*

$$(\text{MP}) \qquad \int \phi_i s(A^*) d\mu = 0, \qquad i = 1, 2, \cdots, n.$$

In order to discuss L_1 approximation let $\mathfrak{L}$ be a linear subspace of the space of integrable functions and suppose $\mathfrak{L}$ is spanned by $\{\phi_i\}$. Set

$$L(\alpha) = \sum_{i=1}^{n} \alpha_i \phi_i$$

and let f be an arbitrary integrable function not in $\mathfrak{L}$. $L(\alpha^*)$ is said to be a *best approximation* to f if

---

Presented to the Society, March 29, 1963 under the title *A moment problem;* received by the editors March 22, 1964.

[1] This author was partially supported by the National Science Foundation under grant NSF G-19915.

[2] This work was done while this author was at the General Motors Research Laboratories.

$$\|L(\alpha^*) - f\| \leqq \|L(\alpha) - f\|$$

for all α.

Suppose that one knows a sign function $s(A)$ for which (MP) holds. One can then interpolate f at n *canonical* points $\{a_i\}$ and, in most but not all cases, obtain a best approximation. This procedure is elaborated upon in [9], [10], [12]. The main result of this paper guarantees the existence of a set of canonical points for any choice of $\{\phi_i(x)\}$. It also has application for nonlinear L_1 approximation [10].

The first connection in which the moment problem arises is as follows: Set

$$Z(\alpha) = \{x \mid f(x) = L(\alpha, x)\}.$$

We have the following

CHARACTERIZATION THEOREM. *A necessary and sufficient condition for $L(\alpha^*)$ to be a best approximation to f is that*

$$\left| \int L(\alpha) \operatorname{sgn} [f - L(\alpha^*)] d\mu \right| \leqq \int_{Z(\alpha^*)} | L(\alpha) | \, d\mu \tag{2}$$

holds for all α.

It is difficult to determine the first correct statement and proof of this result, but it is given in full generality in [3]; for ordinary Lebesgue measure it follows from Theorem 1 of [13]. The hypothesis that μ is nonatomic and $s(A)$ is measurable is not required in this theorem.

There is an interesting special case of this theorem which has sometimes [1], [11] been mistaken for the characterization theorem itself. That is the following

COROLLARY. *If the measure of $Z(\alpha^*)$ is zero, then a necessary and sufficient condition for $L(\alpha^*)$ to be a best approximation to f is that*

$$\int L(\alpha) \operatorname{sgn} [f - L(\alpha^*)] d\mu = 0 \tag{3}$$

for all α.

A secondary connection in which the moment problem arises is with the study of the *Haar Property*. It has recently been shown [7], [8] that no finite dimensional subspace has the Haar Property. Essential to both proofs (which are essentially the same) of this is the establishment of the solution of a certain moment problem by the use of a theorem of Liapounoff [6]. The existence of a solution of (MP), established here, is sufficient for these proofs for $L_1([0, 1], \mu)$.

II. **The theorem and proof.** The proof that the moment problem possesses a solution is outlined as follows. One considers the functions

$$\begin{aligned} \Phi_i(A) &= \int \phi_i s(A) d\mu \\ &= \int_0^{a_1} \phi_i d\mu - \int_{a_1}^{a_2} \phi_i d\mu + \cdots + (-1)^n \int_{a_n}^{1} \phi_i d\mu \end{aligned} \tag{4}$$

for $i=1, 2, \cdots, n$. The functions (4) are continuous functions of A since μ is a finite nonatomic measure. One shows that the domain of definition of each Φ_i may be identified with the n-ball

$$B^n = \left\{ A \,\middle|\, \sum_{i=1}^{n} a_i^2 \leq 1 \right\} .$$

One considers the transformation M of B^n into the $n-1$ sphere

$$S^{n-1} = \left\{ A \,\middle|\, \sum_{i=1}^{n} a_i^2 = 1 \right\}$$

defined by

$$M: A \to \frac{(\Phi_1(A), \Phi_2(A), \cdots, \Phi_n(A))}{\sqrt{\sum [\Phi_i(A)]^2}} . \tag{5}$$

This transformation is a continuous mapping unless all of the $\Phi_i(A)$ are simultaneously zero for some A, i.e., unless the moment problem has a solution. One may show that if A_1 and A_2 are pairs of antipodal points on B^n then

$$\Phi_i(A_1) = -\Phi_i(A_2). \tag{6}$$

Thus M carries pairs of antipodal points of B^n into pairs of antipodal points of S^{n-1}. It is known that such a transformation cannot be continuous.

The remainder of this section contains a detailed exposition of this proof.

LEMMA 1. *The function $\Phi_i(A)$ is a continuous function of A.*

PROOF. This follows from Holder's inequality and some well-known facts from measure theory [2]. In particular if μ is nonatomic then the measure of a finite point set is zero.

The functions $\Phi_i(A)$ are defined in (4) on the n-simplex

$$K_n = \{A \mid 0 \leqq a_1 \leqq \cdots \leqq a_n \leqq 1\}.$$

A specific continuous identification is now constructed which identifies K_n with B^n such that the Φ_i are well defined on B^n and, the essential point, in such a way that if A_1 and A_2 are pairs of antipodal points of B^n then (6) holds.

Thus we inductively construct a mapping ψ_n for each $n \geqq 2$ which has the properties

(a) $\psi_n(0, x_2, \cdots, x_n) = -\psi_n(x_2, \cdots, x_n, 1)$ and each of these points is on the boundary of B^n.

(b) $\Phi_i(\psi_n(A))$ is well defined. That is to say if

$$\psi_n(A) = \psi_n(B) \quad \text{then} \quad \Phi_i(A) = \Phi_i(B).$$

It is easily verified from the specific form (4) of $\Phi_i(A)$ that property (b) follows from the following property

(b′) if $\psi_n(x_1, \cdots, x_n) = \psi_n(y_1, \cdots, y_n)$ then for some k, k of the x_i are equal to k of the y_i and the remaining x_i are equal to each other and the remaining y_i are equal to each other.

We now show ψ_2 explicitly. Set $\psi_2(0, a) = (x, y)$ where $y = 2(a - \frac{1}{2})$ and $x = -\sqrt{(1-y^2)}$. Set $\psi_2(a, 1) = -\psi_2(0, a)$, $\psi_2(\frac{1}{2}, \frac{1}{2}) = (0, 0)$. For the remaining points in K_2, for $0 \leqq t \leqq 1$ set

$$\psi_2[t(\tfrac{1}{2}, \tfrac{1}{2}) + (1-t)(0, \alpha)] = (1-t)\psi_2(0, \alpha),$$
$$\psi_2[t(\tfrac{1}{2}, \tfrac{1}{2}) + (1-t)(a, 1)] = (1-t)\psi_2(a, 1).$$

We note that every point in K_2 is of the form $t(\frac{1}{2}, \frac{1}{2}) + (1-t)(0, a)$ or $t(\frac{1}{2}, \frac{1}{2}) + (1-t)(a, 1)$ for $0 \leqq t \leqq 1$. Further $\psi_2(x_1, x_2) = \psi_2(y_1, y_2)$ if and only if $x_1 = x_2 = \frac{1}{2} - t$, $y_1 = y_2 = \frac{1}{2} + t$. Thus ψ_2 is a mapping of K_2 onto B^2 with properties (a) and (b′).

We define the continuous mapping ψ_{n+1} from ψ_n as follows. Here $|\psi_n(x_1, \cdots, x_n)|$ denotes the usual Euclidean norm of $\psi_n(x_1, \cdots, x_n)$.

$$\begin{aligned} \psi_{n+1}&(0, x_1, \cdots, x_n) \\ &= (-[1 - |\psi_n(x_1, \cdots, x_n)|]^{1/2}, -\psi_n(x_1, \cdots, x_n)), \end{aligned} \tag{7}$$

$$\psi_{n+1}(x_1, \cdots, x_n, 1) = -\psi_{n+1}(0, x_1, \cdots, x_n), \tag{8}$$

$$\psi_{n+1}(\tfrac{1}{2}, \tfrac{1}{2}, \cdots, \tfrac{1}{2}) = (0, 0, \cdots, 0), \tag{9}$$

$$\begin{aligned} \psi_{n+1}(t(\tfrac{1}{2}, \cdots, \tfrac{1}{2}) &+ (1-t)(0, x_1, \cdots, x_n)) \\ &= (1-t)\psi_{n+1}(0, x_1, \cdots, x_n), \quad 0 \leqq t \leqq 1, \end{aligned} \tag{10}$$

$$\begin{aligned} \psi_{n+1}(t(\tfrac{1}{2}, \cdots, \tfrac{1}{2}) &+ (1-t)(x_1, \cdots, x_n, 1)) \\ &= (1-t)\psi_{n+1}(x_1, \cdots, x_n, 1), \quad 0 \leqq t \leqq 1. \end{aligned} \tag{11}$$

Since ψ_n maps K_n *onto* B^n it follows from (7) and (8) that ψ_{n+1} is at least onto the boundary of B^{n+1} and hence it follows from (9), (10) and (11) that ψ_{n+1} maps all of K_{n+1} onto B^{n+1}.

It is clear that ψ_{n+1} is continuous except possibly at the interface between the regions mapped by (10) and (11). Points in that interface are of the form $(a, x_1, \cdots, x_{n-1}, b)$ where $a+b=1$. The mapping is well defined for these points if it is well defined for the points of the interface which map into the boundary of B^{n+1}. These points are of the form $(0, x_1, x_2, \cdots, x_{n-1}, 1)$ and one may verify that (7) and (8) both map this point into the same point of B^{n+1}.

That ψ_{n+1} has property (a) follows directly from (7) and (8).

To establish (b′) suppose that $\psi_{n+1}(x_1, \cdots, x_{n+1}) = \psi_{n+1}(y_1, \cdots, y_{n+1}) = B$. Suppose, for concreteness, that the first coordinate of B is negative. Then ψ_{n+1} is defined by (10). Thus

$$B = (1-t)\psi_{n+1}(0, u_1, \cdots, u_n) = (1-s)\psi_{n+1}(0, v_1, \cdots, v_n)$$

where

$$\begin{aligned}(x_1, \cdots, x_{n+1}) &= t(\tfrac{1}{2}, \cdots, \tfrac{1}{2}) + (1-t)(0, u_1, \cdots, u_n),\\ (y_1, \cdots, y_{n+1}) &= s(\tfrac{1}{2}, \cdots, \tfrac{1}{2}) + (1-s)(0, v_1, \cdots, v_n).\end{aligned} \tag{12}$$

Since $|\psi_{n+1}(0, u_1, \cdots, u_n)| = |\psi_{n+1}(0, v_1, \cdots, v_n)| = 1$ it follows that $s=t$ and $\psi_n(u_1, \cdots, u_n) = \psi_n(v_1, \cdots, v_n)$. By property (b′) for ψ_n we have that k of the u_i equal k of the v_i, while the remaining u_i are equal and the remaining v_i are equal. Since $s=t$ it follows from (12) that ψ_{n+1} has property (b′). A similar argument applies if the first coordinate of B is positive.

The following lemma is well-known [**4**].

LEMMA 3. *There exists no continuous mapping of B^n into S^{n-1} such that pairs of antipodal points are mapped into pairs of antipodal points.*

The main theorem may now be established.

THEOREM. *The moment problem* (MP) *has a solution for any set* $\{\phi_i\}$ *of n integrable functions.*

PROOF. It has been established that the domain K_n, $n \geqq 2$ of definition of $\Phi_i(A)$ may be mapped onto B^n with properties (a) and (b). In particular, a pair A_1, A_2 of antipodal points satisfies (6). The mapping (5) is thus well defined and continuous unless

$$\Phi_i(A) = 0, \qquad i = 1, 2, \cdots, n. \tag{13}$$

Since M takes B^n into S^{n-1} with pairs of antipodal points going into pairs of antipodal points, it follows from Lemma 3 that M is not continuous and hence (13) is satisfied for some A. The proof is trivial if $n=1$. This concludes the proof.

It is of some interest to note [5] that if the ϕ_i are continuous and form a Tchebycheff set, then $s(A)$ is uniquely determined and must have exactly n sign changes.

REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Interscience, New York, 1958; p. 371.

2. P. R. Halmos, *Measure theory*, Van Nostrand, New York, 1950.

3. B. R. Kripke and T. J. Rivlin, *Approximation in the metric of* $L^1(X, \mu)$, Trans. Amer. Math. Soc. **119** (1965), 101–122.

4. S. Lefschetz, *Introduction to topology*, Princeton University Press, Princeton, N. J., 1949; p. 141.

5. P. Laasonen, *Einige Sätze über Tschebyscheffsche Funktionensystems*, Ann. Acad. Sci. Fenn. **52** (1949), 3–24.

6. A. Liapounoff, *Sur les fonctions-vecteurs complètement additives*, Izv. Akad. Nauk SSSR **4** (1940), 465–478.

7. R. M. Moroney, *The Haar problem in* L_1, Proc. Amer. Math. Soc. **12** (1961), 793–795.

8. R. R. Phelps, *Uniqueness of Hahn-Banach extensions and unique best approximations*, Trans. Amer. Math. Soc. **95** (1960), 235–255.

9. J. R. Rice, *The approximation of functions*, Vol. I, Chapter 4, Addison-Wesley, Reading, Mass., 1964.

10. ———, *On the computation of* L_1 *approximations by exponentials, rationals and other functions*, Math. Comp. **18** (1964), 390–396.

11. I. Singer, *Caractérization des éléments de meilleure approximation dans un espace de Banach quelconque*, Acta. Sci. Math. (Szeged) **17** (1956), 181–189.

12. A. F. Timan, *Theory of approximation of functions of a real variable*, Moscow, 1960; English Transl., MacMillan, New York, 1963; Section 2.8.

13. J. L. Walsh and T. S. Motzkin, *Polynomials of best approximation on an interval*, Proc. Nat. Acad. Sci. **45** (1959), 1523–1528.

UNIVERSITY OF WASHINGTON AND
PURDUE UNIVERSITY