UNIQUENESS THEOREMS FOR CONVOLUTION-TYPE EQUATIONS

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Of frequent appearance in the literature are those integral equations which take the form (1), below. These are commonly called convolution-type equations.

$$\text{(1)} \qquad \lambda F(x) - \int_S F(t)K(x-t)\,dt = G(x) \qquad \text{for } x \in S.$$

In particular, when S is the whole real line we obtain the standard convolution equation. Again when S is the half line $(0, \infty)$, we obtain the Wiener-Hopf equation. $S=[0, 1]$ is still another of the classical equations, known to aerodynamicists as the "lifting line equation."

We will be concerned with the uniqueness question for the equation (1), but in the following special sense: We wish to determine conditions on λ and the kernel function K, together with class conditions on K and F, which will insure the uniqueness of the solution of (1) for *all* (measurable) sets S.

For each fixed S, uniqueness is equivalent to the statement:

$$\text{(2)} \qquad \text{If } \lambda F(x) = \int_S F(t)K(x-t)\,dt \text{ for all } x \in S \text{ then } F(x) = 0.$$

Thus, if we redefine F to be 0 in the complement of S, (2) becomes

$$\text{(3)} \qquad \lambda F = F^*K \quad \text{for } x \in S, \qquad F = 0 \quad \text{for } x \notin S \Rightarrow F \equiv 0.$$

And the logical conjunction of the statements (3) for all (measurable) sets S is simply:

$$\text{(4)} \quad \text{If, for each } x, \text{ either } F(x) = 0 \text{ or } \lambda F(x) = (F^*K)(x), \text{ then } F(x) \equiv 0.$$

It is thus our task to find conditions under which (4) holds.

We choose as our setting an arbitrary locally compact abelian group G with Haar measure dt. The class conditions will be $K(t) \in L^1(G)$, $F(t) \in L^\infty(G)$.

Now let $\hat{K}(\xi)$ be the Fourier transform of $K(t)$. As ξ varies through the elements of $\hat{G}$, $\hat{K}(\xi)$ traces out a point set in the complex plane.

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We call this point set C_K, and the closed convex hull of C_K we call H_K.

THEOREM. *Let $K(t)$ have compact support. If $\lambda \notin H_K$ then* (4) *holds.*

Of course, by the previous discussion, this theorem can be restated directly in terms of the original uniqueness question. This restatement can be given as follows:

THEOREM. *Let $K(t)$ have compact support and let $\lambda \notin H_K$. For any (measurable) set S and any function $G(x)$, the integral equation* (1) *has at most one solution.*

We apologize for the onerous restriction to K of compact support. It is undoubtedly an unnecessary one. For G the real line, it can be relaxed to the condition that K falls off exponentially. It may even be that no growth condition whatever on K is needed! The author is simply unable to decide this question.

The condition expressed in this theorem is by no means best possible, but there is a kind of converse in the case of a Hermitian kernel (i.e., one where $K(-t)=K(t)^*$).[1] This converse reads: If (4) holds, $\lambda \neq 0$, and $\hat{G}$ is connected, then $\lambda \notin H_K$.

For proof, note that $\hat{K}(\xi)$ is continuous and real valued so that C_K is a connected real set, and that $H_K=C_K$ or $H_K=C_K \cup 0$. In either case, since $\lambda \neq 0$, $\lambda \in H_K \Rightarrow \lambda \in C_K \Rightarrow \lambda = \int_G X(t)K(t)\,dt$ for some character $X(t)$. If this were so, the choice $F(t)=X(-t)$ would contradict (4).

In other cases the actual necessary and sufficient conditions to insure (4) seem very difficult to obtain. Almost the simplest example, that of the group of the integers, already gives (interesting) trouble. If K consists of a single mass point, then our condition is necessary and sufficient. Suppose, however, that we define our kernel, $K(n)$, as follows:

$K(1)=-2$, $K(2)=-1$, $K(n)=0$ for all other n, and restrict our attention to positive λ.

Accordingly, the condition, $\lambda \notin H_K$, becomes the condition $\lambda > 3/2$. It can be shown, however, that the actual necessary and sufficient condition for (4) is the condition $\lambda > \sqrt{2}$!

From now on we normalize by setting $\lambda = 1$ and before turning to our proof we would like to present the heuristic argument on which it is based. This heuristic argument is actually rigorous in the case of a compact G but is whimsical otherwise since we make use of the "Fourier transform" of $F(t)$.

[1] We use z^* to denote the complex conjugate of z.

We are given that $\bar{F}(x)[F(x)-(F*K)(x)]=0$ identically. Thus $\int_G \bar{F}(x)[F(x)-(F*K)(x)]\,dx=0$. Applying Parseval's theorem we obtain, then, $\int_{\hat{G}}\hat{F}(\xi)^*[\hat{F}(\xi)-\hat{F}(\xi)\hat{K}(\xi)]\,d\xi=0$ (where $\hat{F}(\xi)$ is the Fourier transform of $F(t)$). Hence, unless $\hat{F}(\xi)$ vanishes a.e. (i.e., $F(t)\equiv 0$), we conclude that

$$\frac{\int_{\hat{G}} |\hat{F}(\xi)|^2\hat{K}(\xi)\,d\xi}{\int_{\hat{G}}|\hat{F}(\xi)|^2\,d\xi} = 1,$$

and this shows that $1\in H_K$.

We now turn to the somewhat intricate "truncation" process which supplies the rigor to these heuristic arguments.

Proof of Theorem. We assume that the hypotheses hold, viz., that $1\notin H_K$, that $|F|\leqq 1$, and that $|F|^2=\bar{F}(F*K)$ identically. We can express the first of these conditions by the existence of a complex number α such that

$$\operatorname{Re}[\alpha(1-\hat{K}(\xi))] \geqq 1 \qquad \text{for all } \xi\in\hat{G}. \tag{5}$$

Now let $x_0\in G$ be an arbitrary point and let V be a symmetric compact set containing 0, a neighborhood of x_0, and the support of K.

We denote by V^n, as usual, the set of all $x_1+x_2+\cdots+x_n$ with $x_i\in V$. We now define, for some $n>2$, a function, f, by

$$f = F \text{ in } V^n, \qquad f = 0 \text{ outside } V^n. \tag{6}$$

Now V^n is compact since $+$ is a continuous function on the compact space $V\times V\times V\cdots$. Hence f has compact support and so, since $F\in L^\infty$, we have

$$f\in L^1\cap L^2. \tag{7}$$

Next consider the function $|f|^2-\bar{f}(f*K)$. Clearly this vanishes outside V^n. We claim that it also vanishes inside V^{n-1}. For let $x\in V^{n-1}$, now unless $K(x-t)=0$ it follows that $x-t\in V$ and we conclude that $t\in V^n$, so that $f(t)=F(t)$. Thus $\int_G f(t)K(x-t)\,dt=\int_G F(t)K(x-t)\,dt$ for all $x\in V^{n-1}$. Hence, for these x, $|f|^2-\bar{f}(f*K)=|F|^2-\bar{F}(F*K)=0$ by hypothesis.

If we write $\Delta=V^n-V^{n-1}$, then, we can sum up these remarks in

$$|f|^2-\bar{f}(f*K) = 0 \quad \text{outside } \Delta. \tag{8}$$

Estimating $|f|^2-\bar{f}(f*K)$ *inside* Δ is our next task. Clearly

$$(9)\quad \left|\int_\Delta |f|^2 - \bar{f}(f^*K)\right| \leqq \int_\Delta |F|^2 + \iint_{x\in\Delta,\, t\in V,\, x-t\in V^n} |F(x)|\,|F(x-t)|\,|K(t)|\,dxdt.$$

Note, however, that the conditions $x \notin V^{n-1}$ and $t \in V$ insure that $x-t \notin V^{n-2}$. Combining this with $x-t\in V^n$ insures that $x-t\in V^n - V^{n-2}$. Calling $\Delta' = V^n - V^{n-2}$, then, we may conclude that

$$(10)\quad \iint_{x\in\Delta,\, t\in V,\, x-t\in V^n} |F(x)F(x-t)K(t)|\,dxdt \leqq \iint_{x\in\Delta',\, x-t\in\Delta'} |F(x)F(x-t)K(t)|\,dxdt.$$

For each fixed t, however, Schwarz' inequality tells us that

$$(11)\quad \iint_{x\in\Delta',\, x-t\in\Delta'} |F(x)F(x-t)|\,dx \leqq \int_{\Delta'} |F(x)|^2\,dx$$

and so (10) yields

$$(12)\quad \iint_{x\in\Delta,\, t\in V,\, x-t\in V^n} |F(x)F(x-t)K(t)|\,dxdt \leqq \int_G |K| \cdot \int_{\Delta'} |F|^2.$$

Combining (12) with (9) now yields

$$(13)\quad \left|\int_\Delta |f|^2 - \bar{f}(f^*K)\right| \leqq \left(1 + \int_G |K|\right) \int_{\Delta'} |F|^2.$$

By virtue of (8), however, (13) can be written

$$(14)\quad \left|\int_G |f|^2 - \bar{f}(f^*K)\right| \leqq M \int_{\Delta'} |F|^2, \quad \text{where} \quad M = 1 + \int_G |K|.$$

Parseval's theorem is now justified by (7), and its application gives

$$(15)\quad \left|\int_{\hat{G}} |\hat{f}|^2 (1-\hat{K})\right| \leqq M \int_{\Delta'} |F|^2.$$

Applying (5) to (15) allows the conclusion,

$$(16)\quad \int_{\hat{G}} |\hat{f}|^2 \leqq M\,|\alpha| \int_{\Delta'} |F|^2,$$

and another application of Parseval's theorem yields

$$\int_G |f|^2 = \int_{V^n} |F|^2 \leqq M|\alpha| \int_{\Delta'} |F|^2. \tag{17}$$

Finally (17) may be rewritten

$$\int_{V^{n-2}} |F|^2 \leqq (1-\delta) \int_{V^n} |F|^2, \quad \text{where} \quad \delta = \frac{1}{M|\alpha|}\cdot \tag{18}$$

Repeated application of (18) now gives

$$\int_{V^2} |F|^2 \leqq (1-\delta)^{n-1} \int_{V^{2n}} |F|^2 \leqq (1-\delta)^{n-1} m(V^{2n}) \tag{19}$$

(where m denotes Haar measure).

We now require the following:

LEMMA. *Let V be any compact subset of G. There exist constants c and d such that $m(V^n) \leqq cn^d$, for all n.*

PROOF. Let U be an open set containing V, such that its closure, $\overline{U}$, is compact. Since $\overline{U}^2$ is compact and since it is covered by the totality of translates $x+U$, it follows that $\overline{U}^2 \subset \bigcup_{i=0}^d (x_i+U)$, for some finite collection of x_i. Hence $U^2 \subset \bigcup_{i=0}^d (x_i+U)$ and, by induction, $U^n \subset \bigcup_j (y_j+U)$, where the y_j run through all the possible choices of $x_{i_1}+x_{i_2}+\cdots+x_{i_{n-1}}$. Since the number of such choices is

$$\binom{n+d-1}{d},$$

however, we conclude that

$$m(V^n) \leqq m(U^n) \leqq \binom{n+d-1}{d} m(U).$$

This last expression is $\leqq cn^d$ for an appropriate choice of c, however, and the proof is complete.

If we now combine the result of this lemma with (19) above we conclude that

$$\int_{V^2} |F|^2 \leqq (1-\delta)^{n-1} c(2n)^d, \tag{20}$$

and letting $n \to \infty$ tells us that $\int_{V^2} |F|^2 = 0$, or

$$F = 0 \text{ a.e.,} \qquad \text{in } V^2. \tag{21}$$

But, for $x \in V$, $F*K = \int_{V^2} F(t)K(x-t)\,dt$ and so, by (21), $F*K=0$

throughout V. By hypothesis, however, $|F|^2=\overline{F}(F^*K)$, and so $F=0$ identically throughout V.

In particular, then, F vanishes identically in a neighborhood of x_0. Since x_0 was arbitrary F vanishes identically and the proof is complete.

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DEFINITE AND QUASIDEFINITE SETS OF STOCHASTIC MATRICES[1]

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1. **Introduction.** This note is concerned with asymptotic behaviour of long products of stochastic matrices of a given form. Its objects are:

(a) To prove that a theorem stated (but not proved) by the author in a previously-published paper [2] is equivalent to one proved by Wolfowitz in [1].

(b) To formulate a decision procedure for the above problem, preferable to that given by Wolfowitz in [1].

(c) To solve a related problem.

Familiarity with the above two papers is desirable.

2. **Definitions.** (We adopt here some of the definitions used by Wolfowitz.) A finite square matrix $P=\|p_{ij}\|$ is called stochastic if $p_{ij}\geqq 0$ for all i, j and $\sum_j p_{ij}=1$ for all i.

A stochastic matrix P is called indecomposable and aperiodic (S.I.A.) if

$$Q = \lim_{n\to\infty} P^n$$

exists and all rows of Q are the same. $|P|$ and $\delta(P)$ are defined as

$$|P| = \max_{ij} |p_{ij}|,$$
$$\delta(P) = \max_{j} \max_{i_1 i_2} |p_{i_1 j} - p_{i_2 j}|.$$

With every stochastic matrix P we associate a finite graph having n states (vertices)—n being the order of P—such that transition is

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[1] The note is based on part of the author's D.Sc. thesis submitted to the Senate of the Technion, Israel Institute of Technology, December 1963.