ORDER CONVERGENCE AND TOPOLOGICAL CONVERGENCE

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In a complete lattice it is possible to define a notion of convergence (for arbitrary nets) known as order convergence (o-convergence); for definitions see [**1**, p. 59] and [**3**, p. 65]. As a general rule o-convergence is not a topological convergence; i.e., the lattice cannot be topologized so that nets o-converge if and only if they converge with respect to the topology [**2**]. It is of interest to know when these two types of convergence coincide. A number of questions could be posed here, but we shall deal only with the question of when a topological space can be suitably embedded in a complete lattice. To make this statement more precise we shall give the following definition.

DEFINITION. A topological space is said to be an O-space if it is homeomorphic to a subset Ω_0 of a complete lattice Ω and if every net in Ω_0 converges (with respect to the topology for Ω_0) to a limit in Ω_0 if and only if it o-converges to this limit. For example, every completely regular Hausdorff space is an O-space because it is homeomorphic to a subset of the direct product of unit intervals and if this direct product is partially ordered componentwise, then it becomes a complete lattice in which o-convergence is the same as convergence with respect to the product topology. In this paper we prove the following theorem.

THEOREM. *A topological space is an O-space if and only if it is a regular Hausdorff space.*

PROOF. If X is an O-space, then it is homeomorphic to a subset Ω_0 of a complete lattice Ω, where the topology for Ω_0 has the properties stated in the above definition. Hence, we only need to show that Ω_0 is a regular Hausdorff space. Since limits with respect to o-convergence are unique, Ω_0 is a Hausdorff space. For each nonempty open subset U of Ω_0 define $f(U)=\sup\{\sigma: \sigma\in U\}$ and $g(U)=\inf\{\sigma: \sigma\in U\}$. Then define $S(U)=\{\tau: \tau\in\Omega_0 \text{ and } g(U)\leqq\tau\leqq f(U)\}$. Since Ω_0 is an O-space, $S(U)$ is closed. It is clear that $U\subset S(U)$.

Now let $\sigma_0\in\Omega_0$ be any point. We now define the directed set D to be the collection of all pairs (U, σ), where $U\subset\Omega_0$ is an open neighborhood of σ_0 and $\sigma\in U$. The binary relation $<$ is defined as follows: $(U_1, \sigma_1)<(U_2, \sigma_2)$ iff $U_2\subset U_1$. We will now construct a net in Ω_0 as

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follows: for each $n=(U, \sigma)\in D$ define $\mu_n=\sigma$. It is clear that the net $\{\mu_n: n\in D\}$ converges to σ_0 with respect to the topology for Ω_0, hence, it must also o-converge to σ_0. From the definition of o-convergence we see that $\inf\{f(U): U \in N(\sigma_0)\} = \sigma_0 = \sup\{g(U): U\in N(\sigma_0)\}$, where $N(\sigma_0)$ denotes the collection of all open neighborhoods $U\subset\Omega_0$ of σ_0.

We will use this latter fact to show by contradiction that Ω_0 is regular. If Ω_0 is not regular then there exists a point $\sigma_0\in\Omega_0$ and an open neighborhood V_0 ($V_0\subset\Omega_0$) of σ_0 such that for every open neighborhood U of σ_0 we have $S(U)\cap V_0' \neq\emptyset$. ($V_0'$ denotes the complement of V_0.) Therefore, for each $U\in N(\sigma_0)$ one may select $h(U)\in S(U)\cap V_0'$. Now $\{h(U): U\in N(\sigma_0)\}$ is a net and since $g(U)\leqq h(U)\leqq f(U)$ for all $U\in N(\sigma_0)$, this net must o-converge to σ_0 (recall the results of the previous paragraph). But since $h(U)\in V_0'$ for all $U\in N(\sigma_0)$, this net cannot converge to σ_0 with respect to the topology for Ω_0. This contradicts the fact that Ω_0 is an O-space; hence, Ω_0 must be regular. This completes the proof that every O-space is a regular Hausdorff space.

Now assume that X is a regular Hausdorff space. We may assume that X contains infinitely many points, otherwise all is trivial. Let $\mathfrak{F}$ be the collection of all closed subsets of X which contain at least two points. Now define three sets Ω_-, Ω_0, and Ω_+ as follows: Ω_- and Ω_+ are sets of ordered pairs of the form $(-1, E)$ and $(+1, E)$, respectively, where $E\in\mathfrak{F}$; Ω_0 is the set of ordered pairs of the form $(0, x)$, where $x\in X$. Then define $\Omega=\Omega_-\cup\Omega_0\cup\Omega_+$. The set Ω is partially ordered as follows:

$$\begin{aligned}
(-1, E) &\leqq (-1, F) && \text{iff} \quad F\subset E,\\
(-1, E) &\leqq (0, x) && \text{iff} \quad x\in E,\\
(-1, E) &\leqq (+1, F) && \text{iff} \quad E\cap F\neq\emptyset,\\
(0, x) &\leqq (0, y) && \text{iff} \quad x = y,\\
(0, x) &\leqq (+1, E) && \text{iff} \quad x\in E,\\
(+1, E) &\leqq (+1, F) && \text{iff} \quad E\subset F.
\end{aligned}$$

It is easily shown that $\leqq$ is indeed a partial ordering. It is clear that $(-1, X)$ and $(+1, X)$ are the smallest and largest elements, respectively, in Ω.

In general, Ω is not a lattice, but it can be embedded in a complete lattice $\bar{\Omega}$ (the MacNeille completion); see [**1**, p. 58]. By the embedding, we can regard Ω as a subset of $\bar{\Omega}$. Hence, Ω_0 can be regarded as a subset of $\bar{\Omega}$. We shall topologize Ω_0 so that it is homeomorphic to X; this can be done directly since there is a natural one-to-one correspondence between X and Ω_0.

We will now show that a net in Ω_0 converges with respect to the topology for Ω_0 if and only if it o-converges. Let $\{\sigma_n : n \in D\}$ be a net in Ω_0 which converges to σ with respect to the topology for Ω_0. Since Ω_0 is a regular Hausdorff space, the collection $N(\sigma)$ of closed neighborhoods of σ is a base for the neighborhood system at σ. Now if the singleton $\{\sigma\}$ is open, then there exists $k \in D$ such that $\sigma_n = \sigma$ for all $n > k$; hence, the net o-converges to σ. On the other hand, if the singleton $\{\sigma\}$ is not open, then (putting $\sigma = (0, x)$) we have $\inf\{(+1, E): E \in N(x)\} = \sup\{(-1, E): E \in N(x)\} = (0, x) = \sigma$, where $N(x)$ is the collection of closed neighborhoods of $x \in X$. Hence, for each $E \in N(x)$ there exists $k \in D$ such that $(-1, E) \leqq \sigma_n \leqq (+1, E)$ for all $n > k$. Therefore, the net o-converges to σ.

Now let $\{\sigma_n : n \in D\}$ be a net in Ω_0 which does not converge to σ with respect to the topology for Ω_0. Hence, there must exist a set $E \in \mathfrak{F}$ which does not contain x, where $(0, x) = \sigma$, such that $x_n \in E$ cofinally ($\sigma_n = (0, x_n)$). Hence, $\tau_k = \inf\{\sigma_n : n > k\} \leqq (+1, E)$ for all $k \in D$. Therefore, $\sup\{\tau_k : k \in D\} \leqq (+1, E)$ which means that $\sup\{\tau_k : k \in D\} \neq \sigma$ (recall the definition of the partial order in Ω and the fact that x does not belong to E). This in turn means that the net does not o-converge to σ. Q.E.D.

References

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