4. The table on the preceding page lists the value of $s(n)$ for all $n \leqq 113$. All entries for $s(n)$ were computed individually and checked by means of Theorem 2.

References

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ON THE CONTENT OF POLYNOMIALS

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1. **Introduction.** The content $C(f)$ of a polynomial f with coefficients in the ring R of integers of some algebraic number field K is the ideal in R generated by the set of coefficients of f. This notion plays an important part in the classical theory of algebraic numbers. Answering a question posed to the author by S. K. Stein, we show in the present note that content, as a function on $R[x]$ with values in the set J of ideals of R, is characterized by the following three conditions:

(1) $C(f)$ depends only on the *set* of coefficients of f;

(2) if f is a constant polynomial, say $f(x)=a$, $a\in R$, then $C(f)=(a)$, where (a) denotes the principal ideal generated by a;

(3) $C(f\cdot g)=C(f)\cdot C(g)$ (Theorem of Gauss-Kronecker, see [**1**, p. 105]).

2. **Characterization of content.** Denote by $[f]$ the set of nonzero coefficients of $f\in R[x]$ and call f, g equivalent, of $f\sim g$, if $[f]=[g]$. A polynomial is said to be primitive if its coefficients are rational integers and if the g.c.d. of its coefficients is 1.

Lemma. *Let S be a set of polynomials with coefficients in R and suppose it satisfies*:

(1) $1\in S$;

(2) *if $f\in S$ and $f\sim g$, then $g\in S$;*

(3) *if $f\cdot g\in S$, then $f\in S$ and $g\in S$.*

Then S contains all primitive polynomials.

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PROOF.[1] We will call a polynomial f with rational integer coefficients *special*, if $1\in[f]$ and $a\in f$ implies $-a\in f$. If $p(x)=\sum_{k=0}^{n} c_k x^k$ is primitive, let $q(x)=\sum_{k=0}^{n} d_k x^k$, where $d_0, d_1, \cdots, d_n$ are rational integers such that $\sum_{k=0}^{n} c_k d_{n-k} = 1$. Then $[pq]$ contains 1 and $pq(x^{2n+1}-1)$ is special. By virtue of condition (3) it suffices to show that every special polynomial is in S.

Let therefore f be special and let m_f be the maximum of the absolute values of the coefficients of f. We now proceed by induction on m_f.

If $m_f=1$, then $f\sim x^2-x+1$ and since $(x+1)(x^2-x+1)=x^3+1\sim 1$ and $1\in S$, it follows that $f\in S$.

Let now $m_f=m>1$ and $[f]=\{1,-1,m,-m,a_1,-a_1,\cdots,a_n,-a_n\}$, $|a_k|<m$, $k=1,\cdots,n$. Consider the polynomial $f_1(x)=-1+mx-mx^2+x^3+a_1x^5-a_1x^7+\cdots+a_nx^{4n+1}-a_nx^{4n+3}$. Clearly $f_1\sim f$. Multiplying f_1 by $x+1$ we obtain

$$\begin{aligned} g(x) &= f_1(x)(x+1) \\ &= -1+(m-1)x-(m-1)x^3+x^4+a_1x^5+a_1x^6-a_1x^7-a_1x^8 \\ &\quad +\cdots+a_nx^{4n+1}+a_nx^{4n+2}-a_nx^{4n+3}-a_nx^{4n+4}. \end{aligned}$$

g is special and $m_g=m-1$. Applying the induction hypothesis, we get $g\in S$. Hence $f_1\in S$ by (3) and $f\in S$ by (2), which proves the lemma.

THEOREM. *Let J be the set of ideals in R and h a function on $R[x]$ with values in J satisfying the conditions:*

(1) *if $f, g\in R[x]$ and $f\sim g$, then $h(f)=h(g)$;*
(2) *if f is constant, say $f(x)=a$, $a\in R$, then $h(f)=(a)$;*
(3) *$h(f\cdot g)=h(f)\cdot h(g)$.*

Then $h(f)=C(f)$ for all $f\in R[x]$.

PROOF. Consider first the case, where $1\in[f]$. We may assume f is of the form $x^n+a_1x^{n-1}+\cdots+a_n$, $a_i\in R$, $i=1,\cdots,n$. Let θ_1 be a primitive element of the field K and $\theta_2,\cdots,\theta_r$ its conjugates. Each a_i is then a polynomial $p_i(\theta_1)$ with rational coefficients. Let $a_{ij}=p_i(\theta_j)$, $i=1,\cdots,n$, $j=1,\cdots,r$, and consider $f_j(x)=x^n+a_{1j}x^{n-1}+\cdots+a_{nj}$. Since the coefficients of f are integers of K, the product $F(x)=f_1f_2\cdots f_r$ has rational integer coefficients and those of $f_2\cdots f_r$ are also in R. Now F is primitive as $1\in[F]$. Since the set of all polynomials on which h assumes the value (1) satisfies the conditions of the lemma, we have $h(F)=(1)$ and therefore $h(f)=(1)$.

Next let $C(f)$ be a principal ideal with generating element $a\neq 0$.

[1] For this proof, I am indebted to E. P. Specker.

Then $f(x) = af'(x)$, where $C(f') = (1)$. We can find a polynomial $g'(x) \in R[x]$ such that $1 \in [f' \cdot g']$. Then $h(f'g') = h(f')h(g') = (1)$ and thus also $h(f') = (1)$. Hence $h(f) = h(a)h(f') = (a)(1) = (a) = C(f)$.

If, finally, $C(f)$ is arbitrary, there is a positive integer k such that $(C(f))^k$ is principal (see [1, p. 121]). Now $(C(f))^k = C(f^k) = h(f^k) = (h(f))^k$ and hence $h(f) = C(f)$, because factorization into prime ideals is unique in R. This proves the theorem.

3. **An example.** The Gauss-Kronecker theorem applies to more general rings than just to the rings of integers in a number field. Our theorem however does not remain true if the elements of R are no longer algebraic over the rationals, as will now be shown by an example.

Take for R the ring of polynomials in one indeterminate y and with rational coefficients. R is a principal ideal ring and clearly the Gauss-Kronecker theorem holds for the polynomials of $R[x]$. However, if $f \in R[x]$, say $f(x) = \sum_{j=0}^{n} a_j(y)x^j$, $a_j(y) \in R$, let $m(y) =$ g.c.d. $(a_0(y), \cdots, a_n(y))$ and let d be the degree of f/m with respect to y. Take a fixed but arbitrary nonzero element $r \in R$ and define:

$$h(f) = (m \cdot r^d), \quad \text{if } f \neq 0,$$
$$h(0) = 0.$$

The function h thus defined satisfies the assumptions of the theorem, but clearly $h(f) \neq C(f)$, if f is not a constant polynomial.

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