AN EXACT SEQUENCE IN GALOIS COHOMOLOGY

DOCK SANG RIM[1]

Let A be an integrally closed noetherian domain with the quotient field F. The group of divisors of A is the free abelian group generated by nonzero minimal prime ideals of A and is denoted by $D(A)$. This is canonically isomorphic to the group gotten from the set of all reflexive A-ideals (including fractional ideals) under the rule $\mathfrak{a}\cdot\mathfrak{b} = (\mathfrak{a}\cdot\mathfrak{b})^{**}$ where $\mathfrak{c}^* = \{a\in F \mid a\mathfrak{c}\subset A\} = \mathrm{Hom}_A(\mathfrak{c}, A)$. The divisor class group of A denoted by $C(A)$ is the factor group of $D(A)$ by the principal divisors, i.e. it is defined by the exact sequence[2]

$$0 \to U(F)/U(A) \xrightarrow{D} D(A) \to C(A) \to 0$$

where $D(a) = \sum_{\mathfrak{p}} \nu_{\mathfrak{p}}(a)\mathfrak{p}$ with $(\mathfrak{p}A_{\mathfrak{p}})^{\nu_{\mathfrak{p}}(a)} = aA_{\mathfrak{p}}$. We observe that A is a unique factorization domain if and only if $C(A) = 0$, i.e. if and only if $U(F)/U(A) \to D(A)$ is an isomorphism.

Now let $S\supset R$ be an integral extension of an integrally closed noetherian domain, whose quotient field $L\supset K$ is a separable extension of finite degree. Then we obtain the canonical map $i: D(R)\to D(S)$ given by $\sum_{\mathfrak{p}} \nu_{\mathfrak{p}}\mathfrak{p} \to \sum_{\mathfrak{p}} \nu_{\mathfrak{p}}(\sum_{P\mid\mathfrak{p}} e(\mathfrak{P})\mathfrak{P})$ where $e(\mathfrak{P})$ is the ramification index of $\mathfrak{P}$ in $S\supset R$, i.e. $\mathfrak{p}S_{\mathfrak{P}} = (\mathfrak{P}S_{\mathfrak{P}})^{e(\mathfrak{P})}$. Since the map i sends the principal divisors to principal divisors, it induces the map $i: C(R) \to C(S)$. We denote the kernel of i by $C(S/R)$. Thus $C(S/R)$ is the subgroup of $C(R)$ consisting of those divisor classes which become principal under the extension $S\supset R$. Now let the quotient field extension $L\supset K$ be Galois with the Galois group G. As customary we denote $H^n(G, U(L))$, $H^n(G, U(S))$ by $H^n(L/K)$, $H^n(S/R)$ respectively. The main purpose of this short note is to prove:

THEOREM. *Let $S\supset R$ be an integral extension of an integrally closed noetherian domain whose quotient field extension $L\supset K$ is Galois with the Galois group G. Then we have the exact sequence*

$$\begin{aligned} 0 \to C(S/R) \to H^1(S/R) \to D(S)^G/iD(R) \to C(S)^G/iC(R) \to \\ \to H^2(S/R) \to \bigcap_{\mathfrak{p}} H^2(S_{\mathfrak{p}}/R_{\mathfrak{p}}) \to H^1(G, C(S)) \to H^3(S/R) \end{aligned}$$

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[2] For any commutative ring A, we denote by $U(A)$ the group of invertible elements in A.

where $\bigcap_{\mathfrak{p}} H^2(S_{\mathfrak{p}}/R_{\mathfrak{p}}) = \bigcap_{\mathfrak{p}} \operatorname{Im}(H^2(S_{\mathfrak{p}}/R_{\mathfrak{p}}) \to H^2(L/K))$, $\mathfrak{p}$ *running through all nonzero minimal primes of* R.

REMARK. A somewhat similar exact sequence related to the Brauer groups in the case when $S \supset R$ is unramified was obtained in [2], [3].

PROOF. Firstly we observe that $H^1(G, D(S)) = 0$. Indeed, if we fix, for each nonzero minimal prime $\mathfrak{p}$ in R, a nonzero minimal prime $\mathfrak{P}$ in S lying above $\mathfrak{p}$, and if we denote by $G_{\mathfrak{p}}$ the decomposition subgroup of $\mathfrak{P}$ over $\mathfrak{p}$, then $D(S) \cong \sum_{\mathfrak{p}} Z[G_{\mathfrak{p}}] \otimes_{Z[G]} Z$ as G-modules, where $\mathfrak{p}$ runs through all nonzero minimal primes of R. Consequently $H^*(G, D(S)) = \sum_{\mathfrak{p}} H^*(G_{\mathfrak{p}}, Z)$ and in particular we have $H^1(G, D(S)) = 0$. Now for each minimal prime $\mathfrak{p}$ in R, $S_{\mathfrak{p}}$ is a unique factorization domain and hence $0 \to U(S_{\mathfrak{p}}) \to U(L) \to D(S_{\mathfrak{p}}) \to 0$ is an exact sequence of G-modules. Therefore $0 \to H^2(S_{\mathfrak{p}}/R_{\mathfrak{p}}) \to H^2(L/K) \to H^2(G, D(S_{\mathfrak{p}}))$ is exact and hence we obtain the exact sequence

$$0 \to \bigcap_{\mathfrak{p}} H^2(S_{\mathfrak{p}}/R_{\mathfrak{p}}) \to H^2(L/K) \to H^2(G, D(S)). \tag{1}$$

The exact sequence of G-modules $0 \to U(L)/U(S) \to D(S) \to C(S) \to 0$ together with $H^1(G, D(S)) = 0$ gives us the exact sequences

$$0 \to (U(L)/U(S))^G \to D(S)^G \to C(S)^G \to H^1(G, U(L)/U(S)) \to 0, \tag{2}$$

$$0 \to H^1(G, C(S)) \to H^2(G, U(L)/U(S)) \to H^2(G, D(S)). \tag{3}$$

In turn the exact commutative diagram

$$\begin{array}{ccccccccc} 0 \to & U(K)/U(R) & \to & D(R) & \to & C(R) & \to 0 & & \\ & \downarrow & & \downarrow & & \downarrow & & & \\ 0 \to & (U(L)/U(S))^G & \to & D(S)^G & \to & C(S)^G & \to & H^1(G, U(L)/U(S)) & \to 0 \end{array}$$

yields the exact sequence

$$\begin{aligned} 0 &\to C(S/R) \to (U(L)/U(S))^G/(U(K)/U(R)) \to D(S)^G/iD(R) \\ &\to C(S)^G/iC(R) \to H^1(G, U(L)/U(S)) \to 0. \end{aligned} \tag{4}$$

On the other hand, the exact sequence $0 \to U(S) \to U(L) \to U(L)/U(S) \to 0$ together with Hilbert's Theorem 90 gives us the exact sequences

$$0 \to U(R) \to U(K) \to (U(L)/D(S))^G \to H^1(S/R) \to 0, \tag{5}$$

$$\begin{aligned} 0 &\to H^1(G, U(L)/U(S)) \to H^2(S/R) \to H^2(L/K) \\ &\to H^2(G, U(L)/U(S)) \to H^3(S/R) \to \cdots. \end{aligned} \tag{6}$$

Now the exact commutative diagram

$$
\begin{array}{l}
0 \to H^1(G,\, U(L)/U(S)) \to \\
\qquad\qquad \begin{array}{ccccccc}
H^2(S/R) & \to & H^2(L/K) & \to & H^2(G,\, U(L)/U(S)) & \to & H^3(S/R) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to H^2(G,\, D(S)) & & \longrightarrow & & H^2(G,\, D(S)) & \longrightarrow & 0
\end{array}
\end{array}
$$

together with (1) and (3) yields the exact sequence

$$
\begin{aligned}
(7)\qquad & 0 \to H^1(G,\, U(L)/U(S)) \to H^2(S/R) \to \bigcap_{\mathfrak{p}} H^2(S_{\mathfrak{p}}/R_{\mathfrak{p}}) \\
& \qquad \to H^1(G,\, C(S)) \to H^3(S/R).
\end{aligned}
$$

On the other hand, (4) and (5) gives us

$$
\begin{aligned}
(8)\qquad & 0 \to C(S/R) \to H^1(S/R) \to D(S)^G/iD(R) \to C(S)^G/iC(R) \\
& \qquad \to H^1(G,\, U(L)/U(S)) \to 0.
\end{aligned}
$$

Connecting (7) and (8) we obtain the desired exact sequence.

When $S \supset R$ is unramified, we can relate the 2-dimensional cohomology group with the Brauer group. We denote by $B(S/R)$ the kernel of the canonical map $B(R) \to B(S)$, where $B(\ )$ denotes the Brauer group. Then $H^2(S/R) = B(S/R)$ if $S \supset R$ is unramified and R is a local domain [1] and thus we obtain:

COROLLARY 1. *Let $S \supset R$ be unramified. Then we have the exact sequence*

$$
\begin{aligned}
& 0 \to H^1(S/R) \to C(R) \to C(S)^G \to H^2(S/R) \to \bigcap_{\mathfrak{p}} B(S_{\mathfrak{p}}/R_{\mathfrak{p}}) \\
& \qquad \to H^1(G,\, C(S)) \to H^3(S/R).
\end{aligned}
$$

PROOF. If $S \supset R$ is unramified, then $D(S)^G = iD(R)$ and $\bigcap_{\mathfrak{p}} H^2(S_{\mathfrak{p}}/R_{\mathfrak{p}}) = \bigcap_{\mathfrak{p}} B(S_{\mathfrak{p}}/R_{\mathfrak{p}})$.

If we further assume R to be regular, our exact sequence coincides with the exact sequence in [2], [3].

COROLLARY 2. *Let $S \supset R$ be unramified. If R is regular, we have the exact sequence*

$$
\begin{aligned}
& 0 \to H^1(S/R) \to C(R) \to C(S)^G \to H^2(S/R) \to B(S/R) \\
& \qquad \to H^1(G,\, C(S)) \to H^3(S/R).
\end{aligned}
$$

PROOF. We must show that $\bigcap_{\mathfrak{p}} B(S_{\mathfrak{p}}/R_{\mathfrak{p}}) = B(S/R)$. Since one side inclusion is clear [1], it suffices to show that $\bigcap_{\mathfrak{p}} B(S_{\mathfrak{p}}/R_{\mathfrak{p}}) \subset B(S/R)$, i.e. $\mathrm{Ker}(H^2(L/K) \to H^2(G,\, D(S)) \subset B(S/R)$. Now S, being unramified over a regular domain R, is also regular and hence is a local unique factorization domain. Consequently $D(S)$ is nothing but the group of invertible S-ideals. Let $\alpha \in \operatorname{Ker}(H^2(L/K) \to H^2(G, D(S)))$, and let $\{a_{\sigma,\tau}\}$ be a 2-cocycle representing α. This means that there exists a set $\{A_\sigma\}$ of invertible S-ideals indexed by G such that $a_{\sigma,\tau} A_\sigma A_\tau^\sigma A_{\sigma\tau}^{-1} = S$, i.e. $a_{\sigma,\tau} A_\sigma A_\tau^\sigma = A_{\sigma\tau}$. (We may assume that $A_1 = S$.) Now let Γ be the central simple K-algebra associated with the 2-cocycle $\{a_{\sigma,\tau}\}$, i.e. $\Gamma = \sum_\sigma L u_\sigma$ (direct sum) with the multiplication rule: $(x_\sigma u_\sigma)(x_\tau u_\tau) = x_\sigma x_\tau^\sigma a_{\sigma,\tau} u_{\sigma\tau}$. Then $\Lambda = \sum_\sigma A_\sigma u_\sigma$ which is a subset of Γ is stable under the multiplication since $A_\sigma u_\sigma A_\tau u_\tau = A_\sigma A_\tau^\sigma a_{\sigma,\tau} u_{\sigma\tau} = A_{\sigma\tau} u_{\sigma\tau}$. Thus Λ is an order over R in Γ, and is projective as an R-module. Now consider the canonical map $S \otimes_R \Lambda \to \operatorname{Hom}_S(\Lambda, \Lambda)$ given by $(s \otimes \lambda)(x) = s\chi\lambda$. Since $\alpha \in \operatorname{Ker}(H^2(L/K) \to H^2(G, D(S)) = \bigcap_{\mathfrak{p}} H^2(S_{\mathfrak{p}}/R_{\mathfrak{p}})$, it follows that $S_{\mathfrak{p}} \otimes \Lambda_{\mathfrak{p}} \to \operatorname{Hom}_{S_{\mathfrak{p}}}(\Lambda_{\mathfrak{p}}, \Lambda_{\mathfrak{p}})$ is an isomorphism for all non-zero minimal primes $\mathfrak{p}$ in R. Consequently the canonical map $S \otimes_R \Lambda \to \operatorname{Hom}_S(\Lambda, \Lambda)$ is an isomorphism since both sides are R-projective modules of the same rank. Therefore Λ is an R-separable order in Σ with S as a splitting ring, and this completes our proof.

Bibliography

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Brandeis University