$$\begin{aligned} T(\xi+\eta)x - T(\xi)x &= T(\xi+\eta)x - T(\xi+\eta)x_k + T(\xi+\eta)x_k - T(\xi)x_k \\ &\quad + T(\xi)x_k - T(\xi)x \\ &= T(\xi+\eta-\xi_0)[T(\xi_0)x - T(\xi_0)x_k] \\ &\quad + [T(\xi+\eta)x_k - T(\xi)x_k] \\ &\quad + T(\xi-\xi_0)[T(\xi_0)x_k - T(\xi_0)x] \\ &\in T(\xi+\eta-\xi_0)V_2 + V_2 + T(\xi-\xi_0)V_2 \\ &\subset V_1 + V_1 + V_1 \subset V; \end{aligned}$$

i.e., $[T(\xi+\eta)-T(\xi)]B \subset V$ for $|\eta| < \delta$, $a \leqq \xi+\eta \leqq b$, $\xi_0 < a$. This proves the required continuity of the function $\xi \to T(\xi)$.

That $T(\xi)$ is compact for $\xi > \xi_0$ follows from the fact $T(\xi_0)$ is a compact operator and $T(\xi) = T(\xi-\xi_0)T(\xi_0)$ where $T(\xi-\xi_0)$ is a continuous linear map of E into E.

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Tata Institute of Fundamental Research, Bombay, India

ON RECURSIVELY DEFINED ORTHOGONAL POLYNOMIALS[1]

T. S. CHIHARA

1. **Introduction.** Consider a set $\{P_n(x)\}$ of orthogonal polynomials defined by the classical recurrence formula,

$$\begin{aligned} &P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x) \qquad (n = 1, 2, 3, \cdots), \\ &P_{-1}(x) = 0, \qquad P_0(x) = 1, \qquad c_n \text{ real}, \qquad \lambda_{n+1} > 0. \end{aligned} \tag{1.1}$$

In [2], the author initiated a study of (1.1) based on the chain sequences of Wall [6], the fundamental relation being that the zeros

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of the $P_n(x)$ lie in a subset of $[0, \infty)$ if and only if $c_n>0$ and $\{\lambda_{n+1}/(c_nc_{n+1})\}$ is a chain sequence. This fact can be exploited to derive a number of interesting connections between the behavior of the sequences, $\{c_n\}$ and $\{\lambda_n\}$, and the spectrum of the distribution functions, ψ (support of $d\psi(x)$), with respect to which the $P_n(x)$ are orthogonal.

In the present paper, we wish to continue this study of (1.1), especially in the case where the spectrum is a denumerable set which is bounded below and has no finite point of accumulation. In case the associated Hamburger moment problem (HMP) is indeterminate, however, such conclusions are meaningless in view of the fact that in such a case there exist solutions of the HMP whose spectra have derived sets coinciding with arbitrarily prescribed closed sets [**3**, p. 59]. We therefore also consider the problem of deciding the determinateness of the HMP on the basis of (1.1).

2. **Preliminaries.** We always assume that c_n is real and $\lambda_{n+1}>0$. We then introduce the following notation:

$\{c_n, \lambda_n\}$ denotes the ordered pair $(\{c_n\}, \{\lambda_{n+1}\})$;

$\mathcal{P}\{c_n, \lambda_n\}$ denotes the polynomial set $\{P_n(x)\}$ defined by (1.1);

$\mathcal{S}\{c_n, \lambda_n\}$ denotes the (unique) spectrum of the distribution functions associated with (1.1) whenever the associated HMP is determined. The qualifying phrase, "provided the HMP is determined," is always understood when this notation is used;

$\mathcal{K}$ denotes the set of all pairs $\{c_n, \lambda_n\}$ such that $c_n>0$ and $\{\lambda_{n+1}/(c_nc_{n+1})\}\in\mathcal{C}$, where as in [**2**] $\mathcal{C}$ denotes the set of all chain sequences $\{a_n\}$ such that $0<a_n<1$.

Finally, writing $a_n^{(k)}=a_{n+k}$, we now prove:

Lemma. *If $\{c_n^{(k)}-\epsilon, \lambda_n^{(k)}\}\in\mathcal{K}$ for k sufficiently large, then $\mathcal{S}\{c_n, \lambda_n\}$ has no accumulation points smaller than ϵ.*

If $\{\delta-c_n^{(k)}, \lambda_n^{(k)}\}\in\mathcal{K}$ for k sufficiently large, then $\mathcal{S}\{c_n, \lambda_n\}$ has no accumulation points larger than δ.

Proof. In general, $\mathcal{S}\{c_n-c, \lambda_n\}=\{x-c: x\in\mathcal{S}\{c_n, \lambda_n\}\}$ so that by [**2**, Theorem 1], $\mathcal{S}\{c_n, \lambda_n\}\subset[a, \infty)$ if and only if $\{c_n-a, \lambda_n\}\in\mathcal{K}$. Thus if $\{c_n^{(k)}-\epsilon, \lambda_n^{(k)}\}\in\mathcal{K}$, then $\mathcal{S}\{c_n^{(k)}, \lambda_n^{(k)}\}\subset[\epsilon, \infty)$. Hence by [**2**, Lemma 7], $\mathcal{S}\{c_n, \lambda_n\}$ contains at most k points smaller than ϵ.

Also, $\mathcal{S}\{c-c_n, \lambda_n\}=\{c-x: x\in\mathcal{S}\{c_n, \lambda_n\}\}$ so that $\mathcal{S}\{c_n,\lambda_n\}\subset(-\infty, b]$ if and only if $\{b-c_n, \lambda_n\}\in\mathcal{K}$. Therefore, $\{\delta-c_n^{(k)}, \lambda_n^{(k)}\}\in\mathcal{K}$ implies $\mathcal{S}\{-c_n^{(k)}, \lambda_n^{(k)}\}\subset[-\delta, \infty)$, hence $\mathcal{S}\{-c_n, \lambda_n\}$ contains at most k points smaller than $-\delta$.

Theorem 2.1. *Let $\sigma=\sigma\{c_n, \lambda_n\}$ and $\tau=\tau\{c_n, \lambda_n\}$ denote the smallest* *and largest accumulation points (in the extended real number system) of* $\mathcal{S}\{c_n, \lambda_n\}$. *Then for every* $\{\beta_n\} \in \mathcal{C}$

$$\liminf_{n\to\infty} \{c_n + c_{n+1} - [(c_n - c_{n+1})^2 + 4\lambda_{n+1}/\beta_n]^{1/2}\} \leqq 2\sigma,$$

$$\limsup_{n\to\infty} \{c_n + c_{n+1} + [(c_n - c_{n+1})^2 + 4\lambda_{n+1}/\beta_n]^{1/2}\} \geqq 2\tau.$$

PROOF. Assume $(c_n - \epsilon)(c_{n+1} - \epsilon) > 0$. Then

$$\alpha_n(\epsilon) \equiv \lambda_{n+1}/[(c_n - \epsilon)(c_{n+1} - \epsilon)] \leqq \beta_n$$

if and only if either

$$2\epsilon \leqq c_n + c_{n+1} - [(c_n - c_{n+1})^2 + 4\lambda_{n+1}/\beta_n]^{1/2} \tag{2.1}$$

or

$$2\epsilon \geqq c_n + c_{n+1} + [(c_n - c_{n+1})^2 + 4\lambda_{n+1}/\beta_n]^{1/2}. \tag{2.2}$$

Therefore if (2.1) holds ((2.2) holds) for $n \geqq N$, then by [6, Theorem 20.1] $\{\alpha_n^{(k)}(\epsilon)\} \in \mathcal{C}$ for every $k \geqq N$. Then by the lemma, $\mathcal{S}\{c_n, \lambda_n\}$ has no accumulation points smaller than ϵ (greater than ϵ). That is, $\sigma \geqq \epsilon$ $(\tau \leqq \epsilon)$ for every ϵ satisfying (2.1) ((2.2)) for all n sufficiently large.

COROLLARY. *If* $\{c_n\}$ *and* $\{\lambda_{n+1}\}$ *are bounded, then*

$$c_* - 2\sqrt{\lambda^*} \leqq \sigma \quad \textit{and} \quad \tau \leqq c^* + 2\sqrt{\lambda^*}$$

where $c_* = \liminf_{n\to\infty} c_n$, $c^* = \limsup_{n\to\infty} c_n$, *etc.*

PROOF. This follows from Theorem 2.1 by taking $\beta_n = 1/4$ and using the inequality

$$[(c_n - c_{n+1})^2 + 16\lambda_{n+1}]^{1/2} \leqq |c_n - c_{n+1}| + 4\lambda_{n+1}^{1/2}.$$

If one considers some specific examples (e.g., the Jacobi polynomials), the suggestion appears that in case $\lim_{n\to\infty} c_n = c$ and $\lim_{n\to\infty} \lambda_n = \lambda$, then $\sigma = c - 2\sqrt{\lambda}$ and $\tau = c + 2\sqrt{\lambda}$. This can be proven using relations obtained in [2]. However, this and considerably more has been proven much earlier by O. Blumenthal [1, pp. 16–18] who showed that in the above case, the zeros of the $P_n(x)$ are everywhere dense in $[\sigma, \tau]$. (Blumenthal considers the denominators of "S-fractions" so, technically, his results deal with the case, $\mathcal{S}\{c_n, \lambda_n\} \subset [0, \infty)$. However, the general case is easily reduced to this by a translation and an application of the relations obtained in [2, §2].)

A corollary of Blumenthal's theorem is the fact that if

$$\begin{aligned} &R_n(x) = xR_{n-1}(x) - \gamma_n R_{n-2}(x), \\ (2.3)\quad &R_{-1}(x) = 0, \quad R_0(x) = 1, \quad \gamma_{n+1} > 0, \qquad (n = 1, 2, 3, \cdots), \\ &\lim_{n\to\infty} \gamma_{2n} = \Gamma_0, \quad \lim_{n\to\infty} \gamma_{2n+1} = \Gamma_1, \end{aligned}$$

then $\mathcal{S}\{0, \gamma_n\}$ has no accumulation points on the exterior of $S' = [-\eta, -\zeta] \cup [\zeta, \eta]$, where $\zeta = |\Gamma_0^{1/2} - \Gamma_1^{1/2}|$ and $\eta = \Gamma_0^{1/2} + \Gamma_1^{1/2}$, and the zeros of the $R_n(x)$ are everywhere dense in S'.

This follows from the fact [2, §2] that $R_{2n}(x) = P_n(x^2)$ where $\{P_n(x)\} = \mathcal{P}\{c_n, \gamma_n\}$ with

$$(2.4)\quad c_n = \gamma_{2n-1} + \gamma_{2n}, \lambda_{n+1} = \gamma_{2n}\gamma_{2n+1} \ (n = 1, 2, 3, \cdots; \gamma_1 = 0).$$

Moreover, we also have

$$(2.5)\qquad \gamma_{2n-1} = m_{n-1}c_n \quad \text{and} \quad \gamma_{2n} = (1 - m_{n-1})c_n \quad (n = 1, 2, 3, \cdots)$$

where the m_n are the minimal parameters of $\{\lambda_{n+1}/(c_n c_{n+1})\} \equiv \{\alpha_n\}$. Since $\lim_{n\to\infty}\alpha_n = L = \Gamma_0\Gamma_1(\Gamma_0+\Gamma_1)^{-2} < 1/4$ if $\Gamma_0 \neq \Gamma_1$, we have by [2, Lemma 4]

$$\lim_{n\to\infty} m_n = \tfrac{1}{2}[1 \pm (1 - 4L)^{1/2}],$$

where the + sign is taken if and only if $\{\alpha_n\}$ determines its parameters uniquely. Thus (with $\Gamma_0 \neq \Gamma_1$), $\Gamma_0 < \Gamma_1$ if and only if $\{\alpha_n\}$ determines its parameters uniquely, hence by [2, Theorem 5] if and only if 0 is an isolated point of $\mathcal{S}\{c_n, \lambda_n\}$. That is,

if $\Gamma_0 > \Gamma_1$, then $0 \notin \mathcal{S}\{0, \gamma_n\}$,

if $\Gamma_0 \leqq \Gamma_1$, then $0 \in \mathcal{S}\{0, \gamma_n\}$ and 0 is an accumulation point if and only if $\Gamma_0 = \Gamma_1$.

The preceding—in particular the final assertion—can be nicely illustrated by taking $\gamma_{2n} = a$, $\gamma_{2n+1} = b$ $(n \geqq 1)$. We find from (2.4) that $c_1 = a$, $c_{2n+1} = a + b$ and $\lambda_{n+1} = ab$ $(n \geqq 1)$. For the "kernel polynomials" $\{Q_n(x)\} = \mathcal{P}\{d_n, \nu_n\}$ (see [2, (2.7)]) we have $d_n = a + b$ and $\nu_{n+1} = ab$ $(n \geqq 1)$. It then follows that the $Q_n(x)$ are essentially Tchebicheff polynomials of the first kind:

$$Q_n(x) = (ab)^{n/2}U_n(z), \qquad z = (x - a - b)(4ab)^{-1/2}.$$

The $P_n(x)$ are co-recursive with the $Q_n(x)$ and [3, §5]

$$P_n(x) = Q_n^*(x, c) = (ab)^{n/2}[U_n(z) + (b/a)^{n/2}U_{n-1}(z)], \quad c = -(b/4a)^{1/2}.$$

Also, the $P_n(x)$ are orthogonal with respect to $(\operatorname{sgn} x)d\psi(x)$ where

$$d\psi(x) = x^{-1}(1 - z^2)^{1/2}dx \quad \text{for} \quad (\sqrt{a} - \sqrt{b})^2 \leqq x \leqq (\sqrt{a} + \sqrt{b})^2$$

with $d\psi(x)=0$ otherwise if $|c|\leqq 1/2$ $(b<a)$ while if $|c|>1/2$, then $d\psi(x)=0$ otherwise except for $z=-(a+b)(4ab)^{-1/2}$ $(x=0)$ when ψ has a jump of magnitude $(b-a)/b$.

Since the $R_n(x)$ are orthogonal with respect to $d\psi(x^2)$, we see that $\mathcal{S}\{0, \gamma_n\}$ has the asserted properties.

3. $\sigma\{c_n, \lambda_n\}=\infty$. In [2, Theorem 8] it was proved that if $\lim_{n\to\infty} c_n=\infty$ and $\limsup_{n\to\infty}\lambda_{n+1}/(c_nc_{n+1})=L<1/4$, then $\sigma=\sigma\{c_n,\lambda_n\}=\infty$. The question of whether we can have $\sigma=\infty$ when $L\geqq 1/4$ naturally arises and can be affirmatively answered.

THEOREM 3.1. *If for some* $\{\beta_n\}\in\mathcal{C}$,

$$\lim_{n\to\infty}\frac{c_nc_{n+1}-\lambda_{n+1}/\beta_n}{c_n+c_{n+1}}=\infty,$$

then $\sigma\{c_n,\lambda_n\}=\infty$.

PROOF. With the aid of Bernoulli's inequality, we have

$$\begin{aligned}[(c_n-c_{n+1})^2+4\lambda_{n+1}/\beta_n]^{1/2} &= (c_n+c_{n+1})\left[1+\frac{4\lambda_{n+1}/\beta_n-4c_nc_{n+1}}{(c_n+c_{n+1})^2}\right]^{1/2}\\ &\leqq c_n+c_{n+1}-2\frac{c_nc_{n+1}-\lambda_{n+1}/\beta_n}{c_n+c_{n+1}}.\end{aligned}$$

Thus by Theorem 2.1,

$$\lim_{n\to\infty}\frac{c_nc_{n+1}-\lambda_{n+1}/\beta_n}{c_n+c_{n+1}}\leqq\sigma.$$

The simplest criterion is obtained by taking $\beta_n=1/4$: $\sigma\{c_n,\lambda_n\}=\infty$ if

$$\lim_{n\to\infty}\frac{c_nc_{n+1}-4\lambda_{n+1}}{c_n+c_{n+1}}=\infty. \tag{3.1}$$

For example, if $c_n=n$, $\lambda_{n+1}=(n+1)(n-\sqrt{n})/4$, then (3.1) holds. Moreover, by Carleman's criterion [4, p. 59] (see (4.1)) the associated HMP is determined.

A slightly more general criterion for $\sigma\{c_n,\lambda_n\}=\infty$ is

$$\lim_{n\to\infty}\frac{c_nc_{n+1}}{c_n+c_{n+1}}\left[1-\frac{16n(n+1)\alpha_n}{(2n+1)^2}\right]=\infty,\quad \alpha_n=\frac{\lambda_{n+1}}{c_nc_{n+1}}. \tag{3.2}$$

This can be obtained by taking $\beta_n=(2n+1)^2/[16n(n+1)]$ ($\{\beta_n\}$ has parameters $g_n=(2n+1)/(4n+4)$). In particular, if $\alpha_n\leqq 1/4$,

$$\frac{c_n c_{n+1}}{c_n + c_{n+1}}\left[1 - \frac{16n(n+1)\alpha_n}{(2n+1)^2}\right] \geqq \frac{c_n c_{n+1}}{(c_n + c_{n+1})(2n+1)^2}$$

so (3.2) holds if $c_n = n^2 f(n)$, where $\lim_{n\to\infty} f(n) = \infty$. However, this only requires $4\lambda_{n+1} \leqq n^2(n+1)^2 f(n) f(n+1)$ so that Carleman's criterion may not apply. Thus it may not be clear whether or not we are dealing with a determined HMP. We will consider this question in §4.

We next obtain a simple condition for $\sigma\{c_n, \lambda_n\} = \infty$ which applies in certain cases where $\limsup_{n\to\infty} \alpha_n > 1/4$. In such cases, we must of course have $\liminf_{n\to\infty} \alpha_n < 1/4$ since otherwise $\{c_n + a, \lambda_n\} \notin \mathcal{K}$ for any a [6, p. 79] and we would have $\sigma\{c_n, \lambda_n\} = -\infty$.

Now suppose $\{\alpha_n\} \in \mathcal{C}$ and let $\delta_n = M_n - m_n$, where m_n and M_n are the nth minimal and maximal parameters of $\{\alpha_n\}$. Then it is readily verified that $\alpha_n + \delta_n\delta_{n-1}/4 = (1 - g_{n-1})g_n$, $g_n = (M_n + m_n)/2$. Thus we can take $\beta_n = \alpha_n + \delta_n\delta_{n-1}/4$ in Theorem 3.1 and obtain

$$\frac{c_n c_{n+1} - \lambda_{n+1}/\beta_n}{c_n + c_{n+1}} = \frac{c_n c_{n+1}}{c_n + c_{n+1}}\left(1 - \frac{\alpha_n}{\beta_n}\right) = \frac{c_n c_{n+1}}{c_n + c_{n+1}} \frac{\delta_n \delta_{n-1}}{4\beta_n},$$

so that: $\sigma\{c_n, \lambda_n\} = \infty$ if

$$\lim_{n\to\infty} \frac{\delta_n\delta_{n-1}}{\alpha_n + \delta_n\delta_{n-1}} \frac{c_n c_{n+1}}{c_n + c_{n+1}} = \infty, \qquad \alpha_n = \frac{\lambda_{n+1}}{c_n c_{n+1}}. \tag{3.3}$$

In particular, (3.3) holds if $\{\delta_n\}$ is bounded away from zero and $\lim_{n\to\infty} c_n = \infty$.

For example, consider a periodic sequence $\{a_n\}$ of period p. Suppose $\{a_n\} \in \mathcal{C}$ and let $\{M_n\}$ be its maximal parameter sequence. Then $\{M_n^{(p)}\}$ is the maximal parameter sequence for $\{a_n^{(p)}\} = \{a_n\}$ [2, Lemma 1], hence $\{M_n\}$ is also periodic of period p.

Also, if $\{m_n\}$ is the minimal parameter sequence, then $\{m_n^{(p)}\}$ is a nonminimal parameter sequence for $\{a_n\}$. It follows that $m_n < m_n^{(p)} < M_n^{(p)} = M_n$. Hence

$$\begin{aligned} &\lim_{n\to\infty} m_{np+k} = \mu_k \leqq M_k \qquad (k = 1, 2, \cdots, p), \\ &a_k = (1 - x_{k-1})x_k, \quad 0 < x_k < 1 \qquad (k = 1, 2, \cdots, p;\, x_0 = x_p), \end{aligned} \tag{3.4}$$

where $x_k = \mu_k$ $(k = 1, \cdots, p)$ or $x_k = M_k$ $(k = 1, \cdots, p)$.

Conversely, it is clear that $\{a_n\} \in \mathcal{C}$ if (3.4) is solvable. For the particular case $p = 2$, it is easily verified that (3.4) has distinct solutions (μ_1, μ_2) and (M_1, M_2), $\mu_i < M_i$, if and only if $\sqrt{a_1} + \sqrt{a_2} < 1$ (a unique solution if equality holds). Thus if $\{a_n\}$ is periodic with period 2, then $\{a_n\}\in\mathcal{C}$ and $\delta_n>\min(M_1-\mu_1, M_2-\mu_2)>0$ if and only if $\sqrt{a_1}+\sqrt{a_2}<1$.

Returning to (3.3), we have that $\sigma\{c_n, \lambda_n\}=\infty$ if there exist constants a_1 and a_2 such that $\sqrt{a_1}+\sqrt{a_2}<1$ and $\alpha_{2n+i}\leqq a_i$ $(i=1, 2)$ (and $\lim_{n\to\infty} c_n=\infty$).

4. Determinancy of the HMP. Since the question of the determinancy of the associated HMP is involved in the preceding, it is desirable to have criteria for deciding whether or not the HMP is determined. Most of the well known criteria are expressed directly in terms of the moments (naturally). For our purposes, it is desirable to have criteria expressed in terms of the coefficients in (1.1) as is the case with Carleman's criterion:

$$\text{(4.1)} \qquad \text{if } \sum_{n=1}^{\infty} \lambda_{n+1}^{-1/2} = \infty, \quad \text{the HMP is determined.}$$

We turn to this problem now, restricting ourselves to the case where the moment problem has a solution whose spectrum is a subset of $[0, \infty)$—that is, $\{c_n, \lambda_n\}\in\mathcal{K}$—which is sufficiently general for our purpose.

Therefore assume $\{\alpha_n\}\equiv\{\lambda_{n+1}/(c_nc_{n+1})\}\in\mathcal{C}$ and let m_n denote its minimal parameters. Then there exist positive constants γ_n $(n\geqq 2)$ such that (2.4) and (2.5) hold. Hence, referring to (2.3):

$$P_n(0) = R_{2n}(0) = (-1)^n\gamma_2\gamma_4\cdots\gamma_{2n},$$

$$|p_n(0)|^2 \equiv (\lambda_2\cdots\lambda_{n+1})^{-1}|P_n(0)|^2 = \frac{\gamma_2\gamma_4\cdots\gamma_{2n}}{\gamma_3\gamma_5\cdots\gamma_{2n+1}}$$

$$= \frac{(1-m_1)\cdots(1-m_{n-1})}{m_1\cdots m_{n-1}m_nc_{n+1}}.$$

Referring next to the "numerator polynomials" $\{P_n^{(1)}(x)\}=\mathcal{P}\{c_n^{(1)}, \lambda_n^{(1)}\}$, we must have for the corresponding orthonormal polynomials,

$$|p_n^{(1)}(0)|^2 = \frac{(1-m_{11})\cdots(1-m_{1,n-1})}{m_{11}\cdots m_{1,n-1}m_{1n}c_{n+2}}$$

where the m_{1k} are the minimal parameters of $\{\alpha_n^{(1)}\}$. Now by [2, Lemma 1], $m_{1k}<m_{k+1}$, so that

$$|p_n^{(1)}(0)|^2 > \frac{m_1}{1-m_1}|p_{n+1}(0)|^2.$$

But by a well-known theorem of Hamburger [4, Theorem 2.17], a necessary and sufficient condition for the HMP to be determined is the divergence of at least one of the two series

$$\sum_{n=0}^{\infty} \left| p_n(0) \right|^2 \quad \text{and} \quad \sum_{n=0}^{\infty} \left| p_{n(0)}^{(1)} \right|^2.$$

We therefore have

THEOREM 4.1. *Let* $\{c_n, \lambda_n\} \in \mathcal{K}$. *Then a necessary and sufficient condition for the associated HMP to be determined is the divergence of*

$$\sum_{n=2}^{\infty} \left| p_{n(0)}^{(1)} \right|^2 = \sum_{n=2}^{\infty} \frac{(1-m_{11})(1-m_{12}) \cdots (1-m_{1,n-1})}{m_{11}m_{12} \cdots m_{1,n-1}m_{1n}c_{n+2}}$$

where the m_{1k} *are the minimal parameters for* $\{\alpha_n^{(1)}\}$.

A number of results follow immediately from Theorem 4.1.

THEOREM 4.2. *Let* $\{c_n, \lambda_n\} \in \mathcal{K}$, $\{c_n', \lambda_n'\} \in \mathcal{K}$. *Let* $\lambda_{n+1}'/(c_n'c_{n+1}') \leqq \alpha_n$ $(n \geqq 2)$ *and* $c_n' = O(c_n)$ $(n \to \infty)$. *Then if the HMP associated with* $\{c_n, \lambda_n\}$ *is determined, so is the HMP associated with* $\{c_n', \lambda_n'\}$.

PROOF. Let $\{Q_n(x)\} = \mathcal{P}\{c_n', \lambda_n'\}$ and let m_k' denote the minimal parameters for $\{\lambda_{n+1}'/(c_n'c_{n+1}')\}_{n=2}^{\infty}$. Then $m_k' \leqq m_{1k}$ [6, Theorem 19.6]. It follows that

$$\left| p_n^{(1)}(0) \right|^2 \leqq \frac{c_{n+2}'}{c_{n+2}} \left| q_n^{(1)}(0) \right|^2$$

where $q_n^{(1)}(x)$ is the orthonormal polynomial corresponding to $Q_n^{(1)}(x)$, the numerator polynomial corresponding to $Q_n(x)$.

THEOREM 4.3. *Let* $\{c_n, \lambda_n\} \in \mathcal{K}$ *and assume* $\lim_{n\to\infty} \alpha_n = L$. *Let* $\mu = [1-(1-4L)^{1/2}]/2$. *Then the HMP is indeterminate if*

$$\liminf_{n\to\infty} (c_{n+1}/c_n) > (1-\mu)/\mu,$$

or more generally if $\liminf_{n\to\infty} c_n^{1/n} > (1-\mu)/\mu$. *The HMP is determined if*

$$\limsup_{n\to\infty} (c_{n+1}/c_n) < (1-\mu)/\mu,$$

or more generally if $\limsup_{n\to\infty} c_n^{1/n} < (1-\mu)/\mu$.

PROOF. If $\lim_{n\to\infty} \alpha_n = L$, then [2, Lemma 4] $0 \leqq L \leqq 1/4$ and $\lim_{n\to\infty} m_{1n} = \lim_{n\to\infty} m_n = \mu$. The theorem now follows immediately from Theorem 4.1 by application of the familiar ratio and root tests (in the latter case, using the fact that a convergent sequence and the sequence of its geometric means have the same limit).

Consider, for example, the Stieltjes-Wigert polynomials (see [5, p. 32]) which are known to be associated with an indeterminate moment problem (see [4, p. 22]). Up to a nonzero multiplicative constant, they are the polynomials satisfying (1.1) with

$$c_n = a^{n-1/2}(a^n + a^{n-1} - 1), \qquad \lambda_{n+1} = a^{3n}(a^n - 1), \quad a = q^{-1} > 1.$$

Carleman's criterion does not apply but $\{c_n, \lambda_n\} \in \mathcal{K}$, and

$$\lim_{n\to\infty} \alpha_n = a(a+1)^{-2} \leqq 1/8, \qquad \lim_{n\to\infty} \frac{c_{n+1}}{c_n} = a^2 > a = \frac{1-\mu}{\mu}.$$

Thus by Theorem 4.3, we verify the indeterminacy of the HMP.

We can handle, to a certain extent, cases when $\{\alpha_n\}$ is not convergent by combining Theorems 4.2 and 4.3. We also have the following result.

THEOREM 4.4. *Let $\{c_n, \lambda_n\} \in \mathcal{K}$ and $\alpha_n \leqq 1/4$ $(n \geqq 2)$. Then the HMP is determined if*

$$\sum_{n=1}^{\infty} \frac{n^2}{c_n} = \infty.$$

In particular, if $\alpha_n = 1/4$ $(n \geqq 2)$, the converse is also true.

PROOF. The minimal parameters of the constant sequence $\{1/4\}$ are $k/(2k+2)$ $(k=0, 1, 2, \cdots)$. Thus if $\{P_n(x)\} = \mathcal{P}\{c_n, \lambda_n\}$, where $4\lambda_{n+1} = c_n c_{n+1}$ $(n \geqq 2)$, we have $|p_n^{(1)}(0)|^2 = (n+1)^2/c_{n+2}$. The theorem now follows from Theorems 4.1 and 4.2.

Returning to the example, $c_n = n^2 f(n)$, $\lambda_{n+1} \leqq n^2(n+1)^2 f(n) f(n+1)/4$ $(\lim_{n\to\infty} f(n) = \infty)$, which was considered in §3, we see that the associated HMP will be determined if $\sum_{n=1}^{\infty} [f(n)]^{-1} = \infty$.

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