A REMARK ON THE GENERAL SUMMABILITY THEOREM[1]

T. KAWATA

1. **Introduction.** The following theorem is known as the general summability theorem in the theory of Fourier integrals (see Titchmarsh [4, p. 28]).

THEOREM A. *Suppose that a function $K(x, y, \delta)$ of y belongs to $L^1(-\infty, \infty)$ and satisfies, for a fixed x,*

$$K(x,y,\delta) = O(1/\delta), \quad \text{for} \quad |y-x| \leqq \delta, \tag{1.1}$$

$$= O(\delta^{\alpha}/|x-y|^{1+\alpha}), \quad \text{for} \quad |x-y| > \delta, \tag{1.2}$$

for some positive α and

$$\lim_{\delta\to 0}\int_x^{\infty} K(x,y,\delta)\,dy = 1/2, \tag{1.3}$$

$$\lim_{\delta\to 0}\int_{-\infty}^{x} K(x,y,\delta)\,dy = 1/2. \tag{1.4}$$

Let $f(y)/(1+|y|^{\alpha+1})$ belong to $L^1(-\infty, \infty)$. Then

$$\lim_{\delta\to 0}\int_{-\infty}^{\infty} K(x,y,\delta)f(y)\,dy = (1/2)\{\phi(x)+\psi(x)\}, \tag{1.5}$$

wherever

$$\int_0^h |f(x+t)-\phi(x)|\,dt = O(h) \tag{1.6}$$

and

$$\int_0^h |f(x-t)-\psi(x)|\,dt = O(h) \tag{1.7}$$

as $h\to 0+$.

This implies Fejér's integral theorem and other kinds of ordinary summability theorems. However, Theorem A is not true if $\alpha=0$ and as a matter of fact, it does not imply the Fourier single integral theorem. In this paper we shall prove a theorem which corresponds to the

Received by the editors April 9, 1964.

[1] This work was supported by NSF Grant, GP-1577.

case when $\alpha=0$ and implies as a particular case the ordinary Fourier single integral theorem.

2. **Theorem and its proof.** We shall prove the following

THEOREM. *Suppose that for a fixed x the improper integral $\int_{\to-\infty}^{\to\infty} K(x, y, \delta)dy$ exists. Let for fixed x and $\eta>0$,*

$$K(x, y, \delta) = O(|x-y|^{-1}), \quad \text{for} \quad |x-y| > \eta > 0, \tag{2.1}$$

where O is independent of δ (but may depend on η).

Suppose further that[2]

$$\lim_{\delta\to 0} \int_x^{\to\infty} K(x, y, \delta)\, dy = p, \tag{2.2}$$

$$\lim_{\delta\to 0} \int_{\to-\infty}^{x} K(x, y, \delta)\, dy = 1 - p, \tag{2.3}$$

where $0<p<1$;

$$\lim_{\delta\to 0} \int_\beta^{\to\infty} K(x, y, \delta)\, dy = 0, \quad \text{for every constant } \beta > x, \tag{2.4}$$

$$\lim_{\delta\to 0} \int_{\to-\infty}^{\alpha} K(x, y, \delta)\, dy = 0, \quad \text{for every constant } \alpha < x; \tag{2.5}$$

$$\lim_{\delta\to 0} \int_\alpha^\beta (y-x)K(x, y, \delta)\, dy = 0, \tag{2.6}$$

for every pair α, β of constants such that $x<\alpha<\beta$ or $\alpha<\beta<x$; and

$$\int_x^\beta K(x, y, \delta)\, dy = O(1) \tag{2.7}$$

for every constant β, O being independent of δ.

Let $f(y)/(1+|y|)\in L^1(-\infty, \infty)$ and let $f(y)$ be of bounded variation in a neighborhood of x. Then we have

$$\lim_{\delta\to 0} \int_{-\infty}^{\infty} f(y)K(x, y, \delta)\, dy = pf(x+0) + (1-p)f(x-0). \tag{2.8}$$

Note that the integral on the left-hand side is absolutely convergent, because using (2.1) we have

[2] I missed the conditions (2.4) and (2.5) in the original form of the theorem. I am obliged to Professor S. Bochner for calling this point to my attention.

$$\int_A^\infty |f(y)K(x,y,\delta)|\,dy \leqq \int_A^\infty |f(y)|/|x-y|\,O(1)\,dy$$

for a large A. The same thing is true for $\int_{-\infty}^{-A}$.

PROOF OF THE THEOREM. It is sufficient to prove that

$$\lim_{\delta\to 0}\left\{\int_x^\infty K(x,y,\delta)f(y)\,dy - f(x+0)\int_x^{\to\infty} K(x,y,\delta)\,dy\right\} = 0, \tag{2.9}$$

together with a similar result in terms of $\int_{-\infty}^x$ and $\int_{\to-\infty}^x$.

We may suppose without loss of generality that $f(y)$ is nondecreasing in a right-hand neighborhood of x. For any given positive ϵ we choose η such that

$$|f(y) - f(x+0)| < \epsilon, \qquad \text{for } x \leqq y \leqq x+\eta, \tag{2.10}$$

and we write the expression in (2.9) in the following way.

$$\begin{aligned} I &= \int_x^{x+\eta} K(x,y,\delta)\{f(y) - f(x+0)\}\,dy \\ &\quad + \int_{x+\eta}^\infty K(x,y,\delta)f(y)\,dy - f(x+0)\int_{x+\eta}^{\to\infty} K(x,y,\delta)\,dy \\ &= I_1 + I_2 + I_3, \end{aligned}$$

say.

For every fixed $\eta > 0$, we have, because of (2.4),

$$\lim_{\delta\to 0} I_3 = 0. \tag{2.11}$$

The second mean value theorem shows that

$$|I_1| \leqq \epsilon \int_{x+\xi}^{x+\eta} K(x,y,\delta)dy$$

for some $0<\xi<\eta$. Hence by (2.7), there exists a constant C independent of δ such that

$$|I_1| \leqq C\epsilon. \tag{2.12}$$

In order to handle I_2, we split it into two parts

$$\begin{aligned} I_2 &= \int_{x+\eta}^{x+A} K(x,y,\delta)f(y)\,dy + \int_{x+A}^\infty K(x,y,\delta)f(y)\,dy \\ &= I_{21} + I_{22}, \qquad \eta < A. \end{aligned}$$

Now

$$|I_{22}| \leqq \int_{x+A}^{\infty} \frac{|f(y)|}{y-x}(y-x)\,|K(x,y,\delta)|\,dy \leqq C_1 \int_{x+A}^{\infty} \frac{|f(y)|}{y-x}\,dy,$$

where C_1 is a constant and the integral on the right-hand side may be made as small as desired by taking A large, since $f(y)/(1+|y|) \in L^1(-\infty, \infty)$. Hence we may write

$$|I_{22}| \leqq C_2\epsilon, \tag{2.13}$$

where C_2 is a constant.

Now we choose a step-function $g(y)$ in $(x+\eta, x+A)$, with a finite number of jumps, in such a way that

$$\int_{x+\eta}^{x+A} \left| \frac{f(y)}{y-x} - g(y) \right| dy < \epsilon$$

for fixed A and η. We then write

$$I_{21} = \int_{x+\eta}^{x+A} (y-x)K(x,y,\delta)\left(\frac{f(y)}{y-x} - g(y)\right) dy + \int_{x+\eta}^{x+A} (y-x)K(x,y,\delta)g(y)\,dy.$$

Then

$$|I_{21}| \leqq C \int_{x+\eta}^{x+A} \left| \frac{f(y)}{y-x} - g(y) \right| dy + \left| \int_{x+\eta}^{x+A} g(y)(y-x)K(x,y,\delta)\,dy \right|$$

so that

$$|I_{21}| \leqq C\epsilon + \left| \int_{x+\eta}^{x+A} g(y)(y-x)K(x,y,\delta)\,dy \right|. \tag{2.14}$$

We shall show that the last integral converges to zero as $\delta \to 0$. To do this it is sufficient to prove that

$$\int_{\alpha}^{\beta} (y-x)K(x,y,\delta)\,dy \to 0, \quad \text{for } x+y \leqq \alpha < \beta \leqq x+A \tag{2.15}$$

which is no more than the condition (2.6).

Combining (2.11), (2.12), (2.13), (2.14) and (2.15) we have

$$\limsup_{\delta \to 0} |I| \leqq (2C + C_2)\epsilon.$$

The same thing is true for (2.9) with $(-\infty, x)$ and $f(x-0)$. This proves the theorem.

3. **Remarks.** If $K(x, y, \delta)=\lambda K(\lambda(y-x))$ with $\delta=1/\lambda$ and $K(x)$ is an even function, then (2.8) becomes

$$(3.1)\quad \lim_{\lambda\to\infty}\int_{-\infty}^{\infty} f\left(x+\frac{u}{\lambda}\right)K(u)\,du = \frac{1}{2}\{f(x+0)+f(x-0)\}.$$

The theorem states that (3.1) is true if $f(u)/(1+|u|)\in L^1(-\infty, \infty)$, $f(u)$ is a function of bounded variation in a neighborhood of x and $K(u)$ satisfies the conditions that

$$(3.2)\quad \int_{\to-\infty}^{\to\infty} K(u)\,du = 1,$$

$$(3.3)\quad \lambda\int_{\beta}^{\to\infty} K(\lambda u)\,du \to 0 \quad \text{as} \quad \lambda\to\infty, \quad \text{for every } \beta>0,$$

$$(3.4)\quad \lim_{\lambda\to\infty}\lambda\int_{\alpha}^{\beta} uK(\lambda u)\,du = 0, \quad \text{if} \quad \beta>\alpha>0 \quad \text{or} \quad \alpha<\beta<0,$$

$$(3.5)\quad \int_{0}^{\beta} uK(u)\,du = O(1), \quad \text{for every } \beta,$$

and

$$(3.6)\qquad uK(u) = O\ (1).$$

Since the function $\sin u/(\pi u)$ satisfies above conditions, (3.1) gives us the Fourier single integral theorem. As to the above form (3.1) of summability theorem, one may refer to Bochner [**1**], Bochner-Chandrasekharan [2] and Bochner-Izumi [**3**]. It will be noted that the proof given in this paper is just a generalization of the proof of the Fourier single integral theorem. I_1 and I_3 have been handled in the usual manner. In proving that I_2 converges to zero with the Dirichlet integral, the Riemann-Lebesgue lemma is usually used, but the method employed here is known to be applicable in proving the Riemann-Lebesgue lemma as observed by Bochner and Chandrasekharan [**1**, p. 4].

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Catholic University of America