A PECULIAR BANACH FUNCTION SPACE

G. L. SEEVER[1]

Let (S, Σ, μ) be a σ-finite measure space. A *length function* on (S, Σ, μ) is a non-negative, extended-real-valued function λ on the set $P = P(S, \Sigma, \mu)$ of non-negative, real-valued, Σ-measurable functions on S such that

(1) $\lambda(f) = 0 \Leftrightarrow f = 0$ a.e. (μ);

(2) $\lambda(f+g) \leqq \lambda(f) + \lambda(g)$;

(3) $\lambda(\alpha f) = \alpha\lambda(f)$ for α a non-negative real;

(4) $f_n \uparrow f$ (i.e. $f_n \leqq f_{n+1}$ for all $n \in \omega$ and $\lim_n f_n(s) = f(s)$ for all $s \in S$) $\Rightarrow \lambda(f_n) \uparrow \lambda(f)$;

(5) $E \in \Sigma$, $\mu(E) < \infty \Rightarrow \lambda(\chi_E) < \infty$;

(6) $E \in \Sigma$, $\mu(E) < \infty \Rightarrow \exists \alpha > 0 \ni \int_E f d\mu \leqq \alpha\lambda(f)$ for all $f \in P$.[2]

$\mathfrak{L}_\lambda$ is the set of real-valued Σ-measurable functions f such that $\|f\|_\lambda = \lambda(|f|) < \infty$. $(\mathfrak{L}_\lambda, \|\cdot\|_\lambda)$ is a complete semi-normed linear space (for a proof of this see [1]). $(L_\lambda, \|\cdot\|_\lambda)$ is the corresponding Banach space. L_λ is the *Banach function space*[3] determined by λ. The *associate length function* of λ is the length function λ' defined by

$$\lambda'(f) = \sup\left\{ \int fg d\mu \colon g \in P, \lambda(g) \leqq 1 \right\}, \qquad f \in P,$$

and $L_{\lambda'}$ is the *associate Banach function space* of λ.

Among the Banach function spaces are the familiar L^p spaces and the (less familiar) Orlicz spaces. If $1 < p < \infty$, then the associate Banach function space of L^p is $L^{p'}$ where $1/p + 1/p' = 1$. The associate space of L^1 is L^∞, and that of L^∞ is L^1.[4]

Let λ be a length function on (S, Σ, μ). $f \in L_\lambda$ is *absolutely continuous* iff for any decreasing sequence $\{E_n\}_{n\in\omega}$ in Σ with $\bigcap_{n\in\omega} E_n = \emptyset$, $\lim_n \|f\chi_{E_n}\| = 0$. The set of absolutely continuous elements of L_λ is denoted by $(L_\lambda)^\times$. For example, if $1 \leqq p < \infty$, then $(L^p)^\times = L^p$; $(L^\infty)^\times = \{0\}$. Professor W. A. J. Luxemburg has posed to the author the following question: does there exist a length function λ such that

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[2] This definition of length function doesn't enjoy universal currency. In [3], for example, (5) and (6) are omitted.

[3] We identify a function in $\mathfrak{L}_\lambda$ with the corresponding member of L_λ. Condition (1) states that this identification is the same as identifying functions equal a.e. (μ).

[4] In general, $\lambda'' = \lambda$ for any length function λ. For a proof of this see [1].

$(L_\lambda)^\times=(L_{\lambda'})^\times=\{0\}$? The object of this note is to give an example of such a length function.

Let (S_i, Σ_i, μ_i), $i=1, 2$, be nonatomic measure spaces each of total mass 1, and let $(S, \Sigma, \mu)=(S_1\times S_2, \Sigma_1\otimes\Sigma_2, \mu_1\otimes\mu_2)$. Let λ be defined by

$$\lambda(f) = \int \|f(\cdot, t)\|_\infty d\mu_2(t), \qquad f\in P,$$

where $\|\cdot\|_\infty$ denotes the essential supremum. That the above integrand is measurable follows from

PROPOSITION (LUXEMBURG). *If λ_1 is a length function on (S_1, Σ_1, μ_1), then for $f\in P$, $t\to\lambda_1(f(\cdot, t))$ is Σ_2-measurable.*

PROOF. [2].

One easily verifies that λ is a length function on (S, Σ, μ).

LEMMA 1. *For $f\in P$, $\lambda'(f)=\|\int f(s, \cdot)d\mu_1(s)\|_\infty$.*

PROOF.

$$\begin{aligned}\lambda'(f) &= \sup\left\{\int\int fgd\mu: g\in P,\ \lambda(g)\leqq 1\right\}\\ &\leqq \sup\left\{\int\left[\int f(s,t)\,d\mu_1(s)\right]\|g(\cdot,t)\|_\infty d\mu_2(t): g\in P, \lambda(g)\leqq 1\right\}\\ &\leqq \sup\left\{\left\|\int f(s,\cdot)\,d\mu_1(s)\right\|_\infty \int\|g(\cdot,t)\|_\infty d\mu_2(t): g\in P, \lambda(g)\leqq 1\right\}\\ &\leqq \left\|\int f(s,\cdot)\,d\mu_1(s)\right\|_\infty.\end{aligned}$$

If $g\in P_2=P(S_2, \Sigma_2, \mu_2)$, and if $G\in P$ is defined by $G(s, t)=g(t)$, $(s, t)\in S$, then $\lambda(G)=\int gd\mu_2$. This said, we have

$$\begin{aligned}\lambda'(f) &\geqq \sup\left\{\int\int f(s,t)g(t)\,d\mu_1(s)\,d\mu_2(t): g\in P_2, \int gd\mu_2\leqq 1\right\}\\ &\geqq \sup\left\{\int g(t)\left[\int f(s,t)\,d\mu_1(s)\right]d\mu_2(t): g\in P_2, \int gd\mu_2\leqq 1\right\}\\ &\geqq \left\|\int f(s,\cdot)\,d\mu_1(s)\right\|_\infty.\end{aligned}$$

LEMMA 2. *For $f\in P$, $\lambda(f)\geqq\|\int f(\cdot, t)d\mu_2(t)\|_\infty$.*[5]

[5] This inequality and its use in the proof of the theorem following were pointed out to the author by W. A. J. Luxemburg.

PROOF. If $g \in P_1 = P(S_1, \Sigma_1, \mu_1)$, and if $\int g d\mu_1 \leqq 1$, then $\|f(\cdot, t)\|_\infty \geqq \int f(s, t) g(s) d\mu_1(s)$ for all $t \in S_2$. Therefore

$$\lambda(f) \geqq \sup\left\{\int\left[\int f(s,t)g(s)\, d\mu_1(s)\right] d\mu_2(t) : g \in P_1, \int g d\mu_1 \leqq 1\right\}$$

$$\geqq \sup\left\{\int g(s)\left[\int f(s,t)\, d\mu_2(t)\right] d\mu_1(s) : g \in P_1, \int g d\mu_1 \leqq 1\right\}$$

$$\geqq \left\|\int f(\cdot, t)\, d\mu_2(t)\right\|_\infty .$$

THEOREM. $(L_\lambda)^\times = (L_{\lambda'})^\times = \{0\}$.

PROOF. We first prove that $(L_{\lambda'})^\times = \{0\}$. Since $f \in (L_{\lambda'})^\times \Rightarrow |f| \in (L_{\lambda'})^\times$, it is enough to prove that $f \in L_{\lambda'}, f \geqq 0, f \neq 0 \Rightarrow f \notin (L_{\lambda'})^\times$. Let f be a non-negative nonzero member of $L_{\lambda'}$. There is $r > 0$ such that $F = \{t \in S_2 : \int f(s, t) d\mu_1(s) \geqq r\}$ has positive μ_2-measure. Since μ_2 is non-atomic, there is a decreasing sequence $\{F_n\}_{n\in\omega}$ of Σ_2-measurable subsets of F of positive measure and such that $\mu_2(F_n) \to 0$. Set $E_n = S_1 \times F_n$. $\lambda'(f\chi_{E_n}) = \text{ess. sup}\{\int f(s, t) d\mu_1(s) : t \in S_2\} \geqq r$ for all n, and $\mu(E_n) \to 0$ Therefore, $f \notin (L_{\lambda'})^\times$.

For $f \in P_1$ let $\lambda_0(f) = \|\int f(\cdot, t) d\mu_2(t)\|_\infty$. By Lemma 2, $\lambda(f) \geqq \lambda_0(f)$ for all $f \in P$. Now let $f \in L_\lambda$ be non-negative and nonzero. Interchanging the roles of (S_1, Σ_1, μ_1) and (S_2, Σ_2, μ_2) in the above paragraph yields the existence of an $r > 0$ and a decreasing sequence $\{E_n\}_{n\in\omega}$ of Σ-measurable sets of positive measure such that $\lambda_0(f\chi_{E_n}) \geqq r$ for all n, and $\mu(E_n) \to 0$. Since $\lambda(f\chi_{E_n}) \geqq \lambda_0(f\chi_{E_n})$, we have that $f \notin (L_\lambda)^\times$.

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UNIVERSITY OF CALIFORNIA, LOS ANGELES