STONE-WEIERSTRASS THEOREMS FOR THE STRICT TOPOLOGY

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1. Let X be a locally compact Hausdorff space, E a (real) locally convex, complete, linear topological space, and $\langle C^*(X, E), \beta\rangle$ the locally convex linear space of all bounded continuous functions on X to E topologized with the strict topology β. When E is the real numbers we denote $C^*(X, E)$ by $C^*(X)$ as usual. When E is not the real numbers, $C^*(X, E)$ is not in general an algebra, but it is a module under multiplication by functions in $C^*(X)$.

This paper considers a Stone-Weierstrass theorem for $\langle C^*(X), \beta\rangle$, a generalization of the Stone-Weierstrass theorem for $\langle C^*(X, E), \beta\rangle$, and some of the immediate consequences of these theorems. In the second case (when E is arbitrary) we replace the question of when a subalgebra generated by a subset S of $C^*(X)$ is strictly dense in $C^*(X)$ by the corresponding question for a submodule generated by a subset S of the $C^*(X)$-module $C^*(X, E)$. In what follows the symbols $C_0(X, E)$ and $C_{00}(X, E)$ will denote the subspaces of $C^*(X, E)$ consisting respectively of the set of all functions on X to E which vanish at infinity, and the set of all functions on X to E with compact support.

Recall that the strict topology is defined as follows [2]:

Definition. The strict topology (β) is the locally convex topology defined on $C^*(X, E)$ by the seminorms

$$\|f\|_{\phi,\nu} = \sup_{x\in X} |\phi(x)f(x)|_\nu$$

where ν ranges over the indexed topology on E and ϕ ranges over $C_0(X)$.

When E is the real numbers the above definition may be given simply as: a net $\{f_\alpha : \alpha\in A\}$ converges strictly to a function f iff $\{\phi f_\alpha : \alpha\in A\}$ converges uniformly to ϕf for each ϕ in $C_0(X)$. It is known that $C^*(X, E)$ is complete in the strict topology [2].

2. **Strict approximation in $C^*(X)$.** Stone-Weierstrass theorems were given for $C^*(X)$, under special restrictions by Buck [2]. The exact analogue of the classical theorem (for complex valued functions) was obtained by Glicksberg [3] as a corollary to a version of Bishop's generalized Stone-Weierstrass theorem [1]. The proof is not elementary. We give here a proof which follows quite simply from Buck's

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original theorems. These theorems were proved under the additional hypothesis that either (a) the subalgebra in question contained a function vanishing nowhere or (b) the space X on which the functions were defined was σ-compact.

THEOREM 1. *Let X be a locally compact Hausdorff space and let $\mathfrak{A}$ be a strictly closed subalgebra of $C^*(X)$ which separates points of X and which for each x in X, contains a function g with $g(x) \neq 0$, then $\mathfrak{A} = C^*(X)$.*

PROOF. Let $f \in C^*(X)$ and let $\phi \in C_0(X)$ and $\epsilon > 0$ be given. Let $K(\phi)$ be the σ-compact set outside of which ϕ vanishes identically. $K(\phi)$ may be replaced by a regularly σ-compact subset K of X (i.e. K is a countable union of compact subsets K_n, with K_n contained in the interior of K_{n+1}) which contains $K(\phi)$ and which is an open F_σ set [2]. K is a locally compact Hausdorff space in its relative topology and hence the strict topology is defined for $C^*(K)$, and ϕ is in $C_0(K)$. $\mathfrak{A}$ restricted to K is dense in $C^*(K)$ in the strict topology by (b) above, consequently there is a function g in $\mathfrak{A}$ such that $|\phi(x)f(x) - \phi(x)g(x)| < \epsilon$ for x in K. Since ϕf and ϕg vanish identically outside K, we have $\|g-f\|_\phi < \epsilon$. The conclusion now follows from the fact that $\langle C^*(X), \beta\rangle$ is complete.

The usual theorem for the complex case follows immediately from the above. We also note without proof the following obvious consequences of Theorem 1. (1) If X is a product of locally compact Hausdorff spaces which is locally compact, then every function in $C^*(X)$ may be strictly approximated by finite sums of finite products of functions of one variable on X. (2) There is a 1-1 correspondence between the β-closed ideals of $C^*(X)$ and the closed subsets of X. In particular there is a 1-1 correspondence between the maximal β-closed ideals and the points of X, and each β-closed ideal is the intersection of all β-closed maximal ideals which contain it.

Theorem 1 may be applied to obtain the following result.

THEOREM 2. *If X is a locally compact and σ-compact Hausdorff space, then $\langle C^*(X), \beta\rangle$ is separable iff X is metrizable.*

PROOF. Suppose that X is metrizable, hence since X is σ-compact, X is separable and satisfies the second axiom of countability. Let $\mathfrak{B}$ be a countable base for the topology on X. For each B in $\mathfrak{B}$, the complement of B, being closed, is the zero set of a function f_B in $C^*(X)$. Let $\mathfrak{A}$ be the countable algebra generated by $\{f_B \mid B \in \mathfrak{B}\}$ over the rationals. $\mathfrak{A}$ clearly satisfies the conditions of Theorem 1 and hence is β-dense in $C^*(X)$. Conversely let S be a countable β-dense subset of $C^*(X)$, and let $\mathfrak{A}_S$ be the countable algebra generated by S over the rationals. Let ϕ be any element of $C_0(X)$ which vanishes nowhere on X. Let $\phi\mathfrak{A}_S = \{\phi f \mid f \in \mathfrak{A}_S\} \subseteq C_0(X)$. By the classical Stone-Weierstrass theorem $\phi\mathfrak{A}_S$ is uniformly dense in $C_0(X)$. Let $\mathfrak{A}^*$ be the subset of $\phi\mathfrak{A}_S$ consisting of all functions f such that $\|f\| \leqq 1$. These functions may be used to define a function E on X into the unit cube $Q^\omega = \prod_f \{[0, 1]_f \mid f \in \mathfrak{A}^*\}$ by $E(x)_f = f(x)$. E is easily shown to be a homeomorphism, hence X is metrizable.

It is conjectured that separability for $\langle C^*(X), \beta\rangle$ under the sole hypothesis that X is locally compact and Hausdorff is characterized by the statement: $\langle C^*(X), \beta\rangle$ is separable iff X is metrizable and separable. Note that if X is metrizable and separable then X is necessarily σ-compact.

3. **Strict approximation in** $C^*(X, E)$. Let S be a subset of $C^*(X, E)$ and let x be an element of X, M a closed subspace of E. If $S(x) = \{\alpha \in E \mid \alpha = f(x) \text{ for some } f \in S\}$ is contained in M then we will say S is restricted to M at x. We will call a subset S of $C^*(X, E)$ unrestricted if for each maximal closed subspace M of E and each point x of X, there is a function f in S with $f(x) \notin M$.

THEOREM 3. *Let X be a locally compact Hausdorff space, and E a locally convex, complete, linear topological space. If S is an unrestricted submodule of $C^*(X, E)$, then S is strictly dense in $C^*(X, E)$.*

We first give a proof of the following fact due to Buck [2].

LEMMA. *$C_{00}(X, E)$ is strictly dense in $C^*(X, E)$.*

PROOF. Let $f \in C^*(X, E)$, $\phi \in C_0(X)$, a seminorm ν for E and $\epsilon > 0$ be given. ϕf is in $C_0(X, E)$. Let $K \subseteq X$ be the compact set outside of which $|\phi(x)f(x)|_\nu < \epsilon/2$. There exists a function θ in $C_{00}(X)$ such that $\theta(x) = 1$ for x in K and $\|\theta\| \leqq 1$. θf is in $C_{00}(X, E)$ and $|\phi\theta f(x) - \phi f(x)|_\nu < \epsilon$ for every x in X. Therefore f is in the strict closure of S.

PROOF OF THEOREM 3. In view of the preceding lemma we need only show that if S is an unrestricted submodule, then $C_{00}(X, E)$ lies within the strict closure of S. We show in fact that $C_{00}(X, E)$ is in the uniform closure of S (it is true that $C_0(X, E)$ is in the uniform closure of S).

Let f be any function in $C_{00}(X, E)$ and let a seminorm ρ for E and $\epsilon > 0$ be given. Let $\Psi_\rho(x) = |f(x)|_\rho$. Ψ_ρ is in $C_{00}(X)$. Let K be the compact set outside of which Ψ_ρ vanishes identically. For each g in S, $\Psi_\rho g$ is in S and $\Psi_\rho g(x) = 0$ for x not in K. Let x_0 be any point of K and suppose $f(x_0) = \xi$. By hypothesis there is a function g in S such that $g(x_0) = \xi$. Let $\Psi_\rho(x_0) = a$. If $a \neq 0$ let h_{x_0} be the function $a^{-1}\Psi_\rho g$. $h_{x_0}(x_0)$

$=\xi$, $h_{x_0}(x)=0$ for x not in K, and h_{x_0} is in S. Thus there exists a neighborhood U_{x_0} of x_0 such that $|h_{x_0}(y)-f(y)|_\rho<\epsilon$ for y in U_{x_0}. Suppose that $\Psi_\rho(x_0)=0$. Then let $h_{x_0}=\Psi_\rho g$. $|h_{x_0}(x_0)-f(x_0)|_\rho=|f(x_0)|_\rho=0$ and again by the continuity of seminorms we may choose a neighborhood U_{x_0} of x_0 with $|h_{x_0}(y)-f(y)|_\rho<\epsilon$ for y in U_{x_0}. In the usual manner we obtain a finite number of points x_i in K, of functions h_i in S, and neighborhoods U_i of x_i such that if y is in K then y is in U_i for some i and $|h_i(y)-f(y)|_\rho<\epsilon$. The set of neighborhoods U_i is a point finite open cover of K in its relative topology. Since K is completely regular and compact, there exist functions ϕ_i in $C^*(K)=C(K)$ such that $\phi_i(U_i)\subset[0,1]$, ϕ_i vanishes identically in K outside $U_i\cap K$, and $\sum_i\phi_i(x)=1$ for each x in K. Since K is a compact subset of a completely regular space each ϕ_i may be extended to a function in $C^*(X)$, so that the functions $\phi_i h_i$ are defined for each i and are in S. The function $F=\sum_i\phi_i h_i$ is in S, is identically 0 outside K and $\|F-f\|_\rho<\epsilon$. Thus $C_{00}(X,E)$ lies in the uniform closure of S. From the preceding lemma it follows that S is strictly dense in $C^*(X,E)$.

COROLLARY 1. *Any subspace of $C^*(X,E)$ which is a $C_{00}(X)$ submodule is strictly dense in $C^*(X,E)$. A strictly closed $C_{00}(X)$ submodule is a $C^*(X)$ module.*

COROLLARY 2. *A function f in $C^*(X,E)$ is in the strict closure of the $C^*(X)$ submodule generated by an arbitrary subset S of $C^*(X,E)$ if and only if for any x in X and maximal closed subspace M of E, $S(x)\subseteq M$ implies $f(x)\in M$.*

Theorem 3 generalizes results obtained in [2]. We use the above results to characterize the strictly closed maximal submodules of $C^*(X,E)$.

If M is a maximal closed subspace of E and x is a fixed point of X, let $S_{x,M}=\{f\in C^*(X,E)\,|\,f(x)\in M\}$. Let y be any point of X with $y\neq x$, and $a\in E$, $b\in M$. Let ϕ_1 and ϕ_2 be functions in $C^*(X)$ such that $\phi_1(y)=1$, $\phi_1(x)=0$, and $\phi_2(y)=0$, $\phi_2(x)=1$. Let $h=a\phi_1+b\phi_2$. $h(x)=b$, $h(y)=a$ and h is in $S_{x,M}$. Thus $S_{x,M}(x)=M$ and $S_{x,M}(y)=E$ for each y in X with $y\neq x$. By Corollary 2, $S_{x,M}$ is a strictly closed submodule of $C^*(X,E)$. We show that it is a maximal submodule. Suppose that S' is a submodule of $C^*(X,E)$ such that $S'\supset S_{x,M}$, $S'\neq S_{x,M}$. Then there is a function f in S' such that $f(x)\notin M$. Let $f(x)=\lambda\in E$. E may be represented as a direct sum $M\oplus N$ where N is a complement of M. Then for each $\xi\in E$, $\xi=m+n$, with $m\in M$, $n\in N$, and so for some m_0 in M and n_0 in N, $\lambda=m_0+n_0$. There exists a function g in $S_{x,M}$ such that $g(x)=m_0$. The function $f'=f-g$ in S' has the property $f'(x)=n_0$.

Since M is of deficiency 1 in E each point of N is of the form αn_0, where α is a real number. Thus if ξ is any element of E, $\xi = m + \alpha n_0$, there is a function h in S', $h = g + \alpha f'$ (where g is a function in $S_{x,M}$) such that $h(x) = \xi$, hence $S'(x) = E$. Also $S'(y) = E$ for any point y in X. Now let j be any function in $C^*(X, E)$ and let $j(x) = \eta$. There is a function g_1 in S' with $g_1(x) = \eta$. Let $g_2 = j - g_1$. $g_2(x) = 0$ so g_2 is in $S_{x,M} \subset S'$. $j = g_1 + g_2$ so j is in S' thus S' is $C^*(X, E)$. Consequently $S_{x,M}$ is a strictly closed maximal submodule. Conversely it is easy to see that if S is a strictly closed maximal submodule of $C^*(X, E)$ then S is of the form $S_{x,M}$.

Note that unlike the situation for maximal ideals there may exist maximal submodules which are not strictly closed. Let E be any space (with the usual requirements) which admits a discontinuous linear functional L. Let N be the null space of L. N is a maximal subspace of E and is dense in E. Let x be any point of X and let S be the set of all functions f in $C^*(X, E)$ such that $f(x) \in N$. It is a routine verification that S is a $C^*(X)$ submodule of $C^*(X, E)$. We show that S is a maximal submodule. Let η be any element of E with η not in N and let f be a function in $C^*(X, E)$ with $f(x) = \eta$. If ξ is any point of E then $\xi = \alpha\eta + n$ with $n \in N$ since N is maximal. Let g be any function in $C^*(X, E)$ and let $g(x) = \lambda = \partial\eta + n$, $n \in N$. Let $g_1 = g - \partial f$. $g_1(x) = g(x) - \partial f(x) = n$ so g_1 is in S. $g = g_1 + \partial f$ so S is a maximal submodule. S is not strictly closed, however, by Corollary 2 of Theorem 3. We state the preceding facts as

THEOREM 4. *Let X be a locally compact Hausdorff space, E a locally convex, complete linear space. The strictly closed maximal submodules of $C^*(X, E)$ are the submodules of the form $S_{x,M}$. If E admits no discontinuous linear functional then every maximal submodule is closed.*

THEOREM 5. *Let X be a locally compact Hausdorff space and E a locally convex, complete linear space. A strictly closed submodule S of $C^*(X, E)$ is the intersection of all the maximal strictly closed submodules that contain it.*

PROOF. Let S be any strictly closed submodule of $C^*(X, E)$. Fix x in X and let M_x be the closure of $S(x)$ in E. Let $\mathfrak{M}(x, S)$ be the set of all maximal closed subspaces M of E which contain M_x. $M_x = \bigcap \{ M \mid M \in \mathfrak{M}(x, S) \}$ by the Hahn-Banach theorem for locally convex spaces. Clearly for each x in X, S is contained in $S_{x,M}$ when M is in $\mathfrak{M}(x, S)$. Let $S' = \bigcap \{ S_{x,M} \mid x \in X,\ M \in \mathfrak{M}(x, S) \}$. S' is a strictly closed submodule of $C^*(X, E)$ which contains S and by Corollary 2 of Theorem 3 each function f in S' is in S so that $S' = S$.

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WEAK LIMITS OF POWERS OF A CONTRACTION IN HILBERT SPACE[1]

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Let T be an operator, on the Hilbert space H, with $\|T\| \leqq 1$. Let

$$H_0 = \{x \mid \text{weak lim } T^n x = 0\}, \qquad H_1 = H_0^{\perp}.$$

We shall use the facts, proved in [1], that

1. $x \in H_0$ *if and only if* $\lim (T^n x, x) = 0$ (Theorem 3.1).
2. *On* H_1 *the operator* T *is unitary* (Theorem 1.1).

Given $x \in H$ let $x = x_0 + x_1$, where $x_0 \in H_0$ and $x_1 \in H_1$. The purpose of this note is to find conditions, on the sequence $(T^n x, x)$, that will imply that x_1 is generated by eigenvectors of T. This is related to the notion of mixing for ergodic transformations.

Theorem 1. *Let y be in the subspace generated by $T^n x$, $T^{*n} x$, $n = 1, 2, \cdots$. If* $\lim(T^n y, x) = 0$, *then* weak lim $T^n y = 0$.

Proof. Since H_0 and H_1 are invariant under T, it is enough to prove the theorem for the case when $x \in H_1$ (and thus also $y \in H_1$). If n_i is any subsequence of the integers and $z =$ weak lim $T^{n_i} y$, then z is orthogonal to $T^k x$, $k = 0, \pm 1, \pm 2, \cdots$ (since T is unitary on H_1). But z belongs to the subspace generated by $T^{\pm n} x$. Hence, $z = 0$ and, thus, weak lim $T^n y = 0$.

Corollary. *Let $P(\lambda)$ be a polynomial whose roots of modulus one are $\lambda_1, \cdots, \lambda_k$. If* $\lim(T^n P(T)x, x) = 0$, *then* $x = x_0 + x_1$ *where*: $x_0 \in H_0$, $x_1 = \sum_{i=1}^{k} z_i$, $T z_i = \lambda_i z_i$.

Proof. Let $x = x_0 + x_1$, where $x_0 \in H_0$ and $x_1 \in H_1$. By Theorem 1,

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