FREE GROUPS, HIRSCH-PLOTKIN RADICALS, AND APPLICATIONS TO GEOMETRY

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M. Auslander and R. Lyndon (see [**5**] for a reference) studied properties of $F/[R, R]$, where F is a free group satisfying

$$1 \to R \to F \to f \to 1.$$

Their techniques were homological in nature. Shortly thereafter L. Auslander and M. Kuranishi [1] showed that the above-mentioned results have a geometric interpretation. Recently both H. Neumann (see [**5**] for a reference) and B. H. Neumann [**5**] returned to the study of the paper of Auslander and Lyndon and gave both non-homological proofs and stronger results. The first part of the present paper gives a simple proof of a slightly stronger result (Theorem 2 below) than that given by B. H. Neumann (cf. Theorem 2.7 of [**5**]). The second part extends the geometric interpretation of L. Auslander and M. Kuranishi to this stronger result.

1. **The group theory.** Let R be a normal subgroup of a free group F, let R' denote $[R, R]$, and let bars denote the images of F onto F/R'. We first prove the following two facts.

THEOREM 1. *$\bar{F}$ is torsion-free.*

THEOREM 2. *$\bar{R}$ is the Hirsch-Plotkin radical of $\bar{F}$.*

By the Hirsch-Plotkin radical of a group, we mean its maximal locally nilpotent normal subgroup.

If R^2 also denotes R' and if R^k is defined inductively to be $[R^{k-1}, R]$, then we have the following corollary.

COROLLARY 1. *For each integer $k>1$, F/R^k is torsion-free, and R/R^k is the Hirsch-Plotkin radical of F/R^k.*

PROOF OF COROLLARY 1. Since R/R^k is torsion-free, it follows that if F/R^k is not torsion-free, there is an $f\in F$, $f\notin R$, so that for some integer n, $f^n\in R^k$. But then $\bar{f}^n=1$ contrary to Theorem 1. It is clear that R/R^k is in the Hirsch-Plotkin radical H/R^k of F/R^k. If H/R^k properly contained R/R^k then H/R^2 would properly contain R/R^2 contrary to Theorem 2.

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PROOF OF THEOREM 1. Suppose that $x^n \in R'$ for some $x \in F$ and some positive integer n. Let $S = \mathrm{Sgp}(R, x)$. Then R is normal in S and S/R is cyclic; consequently $R \geqq S'$. On the other hand $S \geqq R$ and hence $S' \geqq R'$. Since $x^n \in R'$, $x^n \in S'$. But S is a free group and hence $x \in S'$ (since S/S' is torsion-free). Thus $x \in R$ since $R \geqq S'$. But $x \in R$ and $x^n \in R'$ imply that $x \in R'$ (since R/R' is torsion-free). Hence $x \in R'$ and $\overline{F}$ is torsion-free as is asserted in Theorem 1.

Before giving the proof of Theorem 2 we define the Schreier property and state a well-known fact as a lemma (cf. [2] for a proof for instance).

The transversal T of the subgroup H of the free group F on a free set of generators X, has the *Schreier property* if whenever $y_1 \cdots y_n$ is a reduced word in T (with each y_i an element of X or its inverse), then $y_1 \cdots y_{n-1}$ is also in T.

LEMMA. *Let X be a free set of generators of a free group F, let H be a nontrivial subgroup of F, and let T be a transversal of H in F with the Schreier property. Then the set of nontrivial elements of $TXT^{-1} \cap H$ is a free set of generators of H.*

PROOF OF THEOREM 2. Let $\overline{H}$ be the Hirsch-Plotkin radical of $\overline{F}$. If $\bar{x} \in \overline{H}$ and $\bar{b} \in \overline{R}$ then for some integer m (depending on $\bar{x}$ and $\bar{b}$), $[\cdots[\bar{b}, \bar{x}], \cdots, \bar{x}] = 1$, the bracket taken m times, or equivalently, if h and x are counter images in F then $[\cdots[b, x], \cdots, x] \in R'$, the bracket taken m times. We shall derive a contradiction to this last relation on the assumption that $\overline{H}$ properly contains $\overline{R}$, or equivalently that the counter image H of $\overline{H}$ properly contains R. Suppose then that $y \in H$, $y \notin R$. Let $x \notin R$ be a member of a free set of generators of $\mathrm{Sgp}(y, R)$ and let $S = \mathrm{Sgp}(x, R)$. Then S is a free subgroup of the free group $\mathrm{Sgp}(y, R)$ and hence by the lemma (since 1 is in the transversal) x is a member of a free set of generators of S. But then each of the other free generators of this set may be modified by a suitable power of x to produce a set of free generators of S consisting of x and a set B of elements b_α which are in R.

Now S/R is cyclic and either of finite order n or of infinite order; in the first case $T_n = \{1, x^{-1}, \cdots, x^{-n+1}\}$ and in the second case $T_\infty = \{1, x, x^{-1}, x^2, x^{-2}, \cdots\}$ is a transversal of R in S with the Schreier property. We use the lemma again to pick a free set of generators of the subgroup R of the free group S. When T_n is the transversal, this set will consist of x^n and the conjugates $b_\alpha^{x^i}$ for $b_\alpha \in B$ and $i = 0, 1, \cdots, n-1$. When T_∞ is the transversal, it will consist of $b_\alpha^{x^i}$ for $b_\alpha \in B$ and all integers i.

We will now have our contradiction when we show that if b is one of the b_α then $[\cdots[b, x], \cdots, x]$, the bracket taken m times, is not in R' for each natural number m. Indeed $[b, x]=b^{-1}b^x$; $[[b, x], x]=b^{-x}b(b^{-1}b^x)^x$ which is congruent to $b^{x^2}(b^{-x})^2b$ mod R'. A direct induction then gives that $[\cdots[b, x], \cdots, x]$, the bracket taken m times, is congruent to

$$\prod_{i=0}^{m} (b^{x^{m-i}})^{(-1)^i\binom{m}{i}} \bmod R'.$$

When T_∞ is the transversal (and also when T_n is the transversal with $m=1$), the elements $b^{x^{m-i}}$ are free generators of R, and hence the product in the last expression is obviously not in R'. In case T_n is the transversal with $m>1$, we let ϵ denote a primitive nth root of 1 and note that $(1-\epsilon)^m \neq 0$. We expand this expression by the binomial theorem and combine like powers of ϵ after using only the relation $\epsilon^n=1$ wherever possible. Since $(1-\epsilon)^m\neq 0$, it follows that it is not possible for all the coefficients of the ϵ_i to be zero after the combinations described. Looking back at the product describing

$$[\cdots[b, x], \cdots, x] \bmod R',$$

the bracket taken m times, we see that this product cannot be in R' as was to be shown. This proves the theorem.

2. **The applications to geometry.** Let Γ be a torsion-free finitely generated group with a nilpotent Hirsch-Plotkin radical N so that $f=\Gamma/N$ is finite. We will call such a group Γ a nil-admissible group; thus a nil-admissible group Γ satisfies the diagram

$$1 \to N \to \Gamma \to f \to 1$$

with f a finite group. By a theorem of Mal'cev [3] there is a unique connected, simply connected, nilpotent Lie group $L(N)$ with the following properties:

(1) $N \leqq L(N)$.

(2) $L(N)/N$ is compact.

(3) Every automorphism of N is uniquely extendable to $L(N)$.

Now it is straightforward to verify that if we have an exact sequence of groups

$$1 \to A \to B \to C \to 1$$

and A is contained in a group A^* with the property that every automorphism of A is uniquely extendable to an automorphism of A^*, then there exists a unique B^* satisfying the diagram

$$\begin{array}{ccccccccc} 1 & \to & A & \to & B & \to & C & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & A^* & \to & B^* & \to & C & \to & 1 \end{array}$$

where all vertical arrows are inclusions. Hence we have a group Γ^* satisfying the diagram

$$\begin{array}{ccccccccc} 1 & \to & N & \to & \Gamma & \to & f & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & L(N) & \to & \Gamma^* & \to & f & \to & 1 \end{array}$$

where the vertical arrows are inclusions. But Γ^* is now a split extension; i.e., $\Gamma^* = L(N)\cdot f$. This follows directly from the divisibility of $L(N)$ or from a result of Mostow [4]. Hence $\Gamma^* \leqq L(N)\cdot A(L(N))$, where $A(L(N))$ denotes the group of continuous automorphisms of $L(N)$. But we can define $L(N)\cdot A(L(N))$ as a group of transformations of $L(N)$ by first defining the map ψ so that $\psi(n, g) = n$ and then defining $(n, g)m = \psi((n, g)(m, e))$ where $n, m \in L(N)$, $g \in A(L(N))$ and e is the identity element of $A(L(N))$. We will call this the affine group of the nilpotent Lie group $L(N)$. Thus Γ^* has been interpreted as a subgroup of affinities of $L(N)$. Since:

(1) Γ/N is finite,

(2) $L(N)/N$ is a compact manifold,

(3) Γ is torsion-free,

it is straightforward to verify that Γ operates properly discontinuously on $L(N)$ and with compact fundamental domain. We have therefore proven the following theorem.

THEOREM 3. *If* Γ *is nil-admissible then* Γ *operates properly discontinuously and with compact fundamental domain on a topological euclidean space.*

COROLLARY 2. *Let G be any module on which* Γ *operates trivially. Then the cohomology groups vanish,* $H^q(\Gamma, G) = 0$ *for all* $q > \dim L(N)$.

In view of Corollary 1, when a normal subgroup R has finite index in a finitely generated free group F, then F/R^k is nil-admissible and hence Theorem 3 is directly applicable to $\Gamma = F/R^k$.

Added in proof. It has recently come to our attention that some of our results are closely related to results already in the literature. Thus Theorem 1 is due to G. Higman and is also a very special case of Theorem 5 in Baumslag, *Wreath products and extensions*, Math. Z. **81** (1963), 286–289. Theorem 2 is very closely related to Lemma 3 of Gruenberg, *The residual nilpotence of certain presentations of finite groups*, Arch. Math. **13** (1962), 408–417.

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INJECTIVE MODULES UNDER CHANGE OF RINGS[1]

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Introduction. Let U and V be rings with unit element and $k: U \to V$ a ring epimorphism with kernel I. Every V-module (all modules are left modules) can be regarded as a U-module under k and hence it makes sense to ask when a V-module is U-injective.

For $u \in U$, we denote the left ideal $\{c \mid c \in U,\ cu = 0\}$ by $0:u$. The answer to the above question is simply:

Criterion. A V-module A is U-injective if and only if it satisfies the following two conditions:

(1) *A is V-injective*;

(2) *If $u \in I$ and $a \in A$ and* $(0:u)a = 0$, *then* $a = 0$. (The cutest way to put (2) is $0:(0:u) = 0$ for all $u \in I$.)

We prove the criterion in §1 and make an application of it to G-modules in §2. (G stands for a finite group.)

1. **Proof of the criterion.** Let the V-module A be U-injective. In order to prove condition 1, we select a left ideal M of V and a V-homomorphism $g: M \to A$. We have to produce an element $a \in A$ such that $g(v) = va$ for all $v \in M$. (See [**1**, Theorem 3.2, p. 8].) Hereto we consider the left ideal $k^{-1}(M)$ of U and the U-homomorphism $gk: k^{-1}(M) \to A$. Since A is U-injective, there exists an $a \in A$ such that $gk(u) = ua$ for all $u \in k^{-1}(M)$. Let now $v \in M$. Since k is an epi, $v = k(u)$ for some $u \in k^{-1}(M)$ and, hence, $g(v) = gk(u) = ua$. The action of U on A is such that $ua = k(u)a = va$ and condition 1 is proved.

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