# SECOND ORDER LINEAR AND NONLINEAR DIFFERENTIAL EQUATIONS[1]

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1. **Introduction.** It was noted by Pinney [2] that the solution of the nonlinear differential equation $y''+p(x)y'+cy^{-3}=0$, $c$ constant, can be written in the form $y=(u_1^2-u_2^2)^{1/2}$, where $u_1(x)$, $u_2(x)$ are appropriately chosen solutions of the linear equation $u''+p(x)u=0$. This result led Thomas [3] to ask: What equations of order $n$ have general solutions expressible in the form $y=F(u_1, \cdots, u_n)$, where $u_1, \cdots, u_n$ constitute a variable set of solutions of a linear equation? Thomas answered this question when the underlying linear equation is of the first order, $u'+pu=q$. He also gave the answer for homogeneous second order equations, $u''+pu'+qu=0$, when $F$ depends only on one $u$, or when $F$ is homogeneous of nonzero degree in two $u$'s. Using the theory of passive partial differential equations, Herbst [1] removed these restrictions, obtaining the following general theorem.

THEOREM 1. *If $u_1(x)$, $u_2(x)$ are variable independent solutions with Wronskian $w$ of the linear equation*

$$u'' = w'w^{-1}u' + qu, \tag{1.1}$$

*where $w(x)$, $q(x)$ are arbitrarily prescribed functions, then the equation*

$$y'' = w'w^{-1}y' + f(y, y', w, q) \tag{1.2}$$

*has general solution $y=F(u_1, u_2)$ if, and only if, $f$ has the form*

$$f = qZ(y) + A(y)(y')^2 + C(y)w^2, \tag{1.3}$$

*where $Z$, $A$, $C$ satisfy*

$$Z' - AZ = 1, \qquad ZC' + (3 - AZ)C = 0. \tag{1.4}$$

Herbst's theorem determines the form of $f$. In Herbst's analysis, however, $F$ is determined only as a solution of a system of four partial differential equations. The purpose of this paper is to give a simple characterization of $F$. On the basis of information obtained a method is developed for the solution of (1.2) when $f$ has form (1.3) and $Z$, $A$, $C$ satisfy (1.4).

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2. **Main result.** For the determination of $F$ only a limited class of linear equations (1.1) is needed, namely, the class for which $w$ and $q$ are constants, $w\neq 0$. Our main result is as follows.

THEOREM 2. *Suppose that* $f(y, y', w, q) \in C'$ *on a domain* $R=\{(y, y', w, q) \mid m<y<M\}$, *that*

$$\xi(y) \equiv f(y, 0, 0, 1) \neq 0, \qquad m < y < M, \tag{2.1}$$

*and that* $F(u_1, u_2)\in C''$ *and* $m<F<M$ *on a domain* $V$. *Suppose further that for arbitrary constants* $w\neq 0$ *and* $q$, *if* $u_1(x)$, $u_2(x)$ *are solutions of* (1.1) *with Wronskian* $w$ *such that* $(u_1, u_2)\in V$ *for* $x$ *on an interval* $I$, *then* $y=F(u_1, u_2)$ *satisfies* (1.2) *on* $I$. *Under these conditions, if* $m<\eta<M$, *then* $F$ *can be written*

$$F = \Phi(\omega^{1/2}), \qquad (u_1, u_2) \in V, \tag{2.2}$$

*where* $\omega(u_1, u_2)$ *is a homogeneous polynomial of degree* 2, *positive in* $V$, *and* $\Phi$ *is the inverse of*

$$\phi(y) = \exp\left\{\int_\eta^y \xi^{-1}(t)\,dt\right\}, \qquad m < y < M. \tag{2.3}$$

The proof is based on the uniqueness property of solutions of differential equations. Knowledge of the form (1.3) of $f$ is not required. We observe that in the Pinney case $\omega=u_1^2-u_2^2$ and $\Phi(y)=y$.

PROOF. We show first that

$$L(u_1, u_2) \equiv u_1F_1(u_1, u_2) + u_2F_2(u_1, u_2) = \xi(F(u_1, u_2)),\ (u_1, u_2) \in V, \tag{2.4}$$

where $F_i=\partial F/\partial u_i$, $i=1, 2$. Let $(v_1, v_2)$ be a point of $V$ other than the origin. Let $v_1'$, $v_2'$ be arbitrarily chosen so that $w=v_1v_2'-v_2v_1'\neq 0$. Let $u_1(x)$, $u_2(x)$ be the solution on $(-\infty, \infty)$ of $u''=u$ with initial values $u_i(0)=v_i$, $u_i'(0)=v_i'$. Then $u_1$, $u_2$ have Wronskian $w$, and $(u_1, u_2)\in V$ for $x$ on an interval $I$ containing $x=0$. Hence, by hypothesis, $y=F(u_1, u_2)$ satisfies $y''=f(y, y', w, 1)$ on $I$. Differentiating $y$ twice, and placing $x=0$ in the final equation, we get

$$\begin{aligned} v_1F_1 + v_2F_2 + F_{11}(v_1')^2 + 2F_{12}v_1'v_2' + F_{22}(v_2')^2 \\ = f(F, v_1'F_1 + v_2'F_2, w, 1), \end{aligned} \tag{2.5}$$

where $F$, $F_i$, $F_{ij}$ are evaluated at $(v_1, v_2)$. With $v_1$, $v_2$ fixed we can let $v_1', v_2'\to 0$ in (2.5), to obtain $L(v_1, v_2)=f(F(v_1, v_2), 0, 0, 1)$. Hence, (2.4) holds save possibly at the origin. We observe, however, that the origin cannot be a point of $V$. Otherwise, by continuity, we would have $\xi(F(0, 0))=L(0, 0)=0$, contrary to (2.1).

Now let $P_0$: $(v_1^0, v_2^0)$ be a fixed point in $V$. Let $V_0$ be an open disk in

$V$ having center at $P_0$. We prove that there exist constants $a_0$, $b_0$, $c_0$, not all zero, such that

$$(2.6)\quad (b_0u_1+c_0u_2)F_1(u_1, u_2)-(a_0u_1+b_0u_2)F_2(u_1, u_2)=0, \quad (u_1, u_2)\in V_0.$$

By (2.1) and (2.4), the gradient of $F$ is different from zero at $P_0$. Accordingly, there are functions $\lambda_i(s)$, $i=1, 2$, of class $C''$ on an interval $|s|<\sigma$, where $0<\sigma$, such that $\lambda_i(0)=v_i^0$, $0<(\lambda_1'(0))^2+(\lambda_2'(0))^2$, and

$$(2.7)\quad (\lambda_1, \lambda_2)\in V_0, \qquad F(\lambda_1, \lambda_2) = F(\overset{0}{v}_1, \overset{0}{v}_2), \qquad |s| < \sigma.$$

Let $(v_1, v_2)$ be a second point of $V_0$ such that $w=v_1^0v_2-v_2^0v_1\neq 0$. Since $L(\lambda_1(s), \lambda_2(s))\neq 0$, the equation

$$(2.8)\quad \begin{pmatrix} F_1(\lambda_1, \lambda_2) & F_2(\lambda_1, \lambda_2) \\ -\lambda_2 & \lambda_1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} F_1(\overset{0}{v}_1, \overset{0}{v}_2) & F_2(\overset{0}{v}_1, \overset{0}{v}_2) \\ -\overset{0}{v}_2 & \overset{0}{v}_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

admits for each $s$ on $(-\sigma, \sigma)$, a unique solution $z_i=z_i(s)$. From (2.4) and (2.7) we have

$$(2.9)\qquad L(\lambda_1, \lambda_2) = L(\overset{0}{v}_1, \overset{0}{v}_2), \qquad |s| < \sigma.$$

Hence,

$$(2.10)\qquad \begin{pmatrix} z_1(s) \\ z_2(s) \end{pmatrix} = \begin{pmatrix} \alpha(s) & \beta(s) \\ \gamma(s) & \delta(s) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

where

$$(2.11)\quad L(\overset{0}{v}_1, \overset{0}{v}_2)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \lambda_1 & -F_2(\lambda_1, \lambda_2) \\ \lambda_2 & F_1(\lambda_1, \lambda_2) \end{pmatrix}\begin{pmatrix} F_1(\overset{0}{v}_1, \overset{0}{v}_2) & F_2(\overset{0}{v}_1, \overset{0}{v}_2) \\ -\overset{0}{v}_2 & \overset{0}{v}_1 \end{pmatrix}.$$

We note that $z_i=v_i$ for $s=0$. Hence, for $|s|$ sufficiently small, $|s|<\sigma_1$, say, where $0<\sigma_1\leqq\sigma$, we have $(z_1(s), z_2(s))\in V_0$.

Suppose for the moment $s$ held fast on $(-\sigma_1, \sigma_1)$, and consider the linear functions $u_i(x)=u_i(x, s)=(1-x)\lambda_i+xz_i$, $0\leqq x\leqq 1$. We have $u_i(0)=\lambda_i$, $u_i(1)=z_i$. Since $V_0$ is convex it follows that $(u_1, u_2)\in V_0$, $0\leqq x\leqq 1$. Further, $u_1$, $u_2$ satisfy $u''=0$ and, by (2.8), have Wronskian

$$\lambda_1(z_2 - \lambda_2) - \lambda_2(z_1 - \lambda_1) = \lambda_1 z_2 - \lambda_2 z_1 = \overset{0}{v}_1 v_2 - \overset{0}{v}_2 v_1.$$

Hence, $y=y(x)=y(x, s)=F(u_1(x, s), u_2(x, s))$ satisfies

$$(2.12)\qquad y'' = f(y, y', w, 0), \qquad 0 \leqq x \leqq 1.$$

The equation (2.12) is independent of $s$. Referring to (2.7), (2.8), and (2.9), we verify that

$$
\begin{aligned}
y(0, s) &= F(\lambda_1(s), \lambda_2(s)) = F(\overset{0}{v}_1, \overset{0}{v}_2),\\
y'(0, s) &= (z_1 - \lambda_1)F_1(\lambda_1, \lambda_2) + (z_2 - \lambda_2)F_2(\lambda_1, \lambda_2)\\
&= v_1F_1(\overset{0}{v}_1, \overset{0}{v}_2) + v_2F_2(\overset{0}{v}_1, \overset{0}{v}_2) - L(\overset{0}{v}_1, \overset{0}{v}_2).
\end{aligned}
\tag{2.13}
$$

Thus, $y(x, s)$, $y'(x, s)$ have the same initial values, $y(0, s)$, $y'(0, s)$, independently of $s$. Since $f \in C'$ on $R$, we conclude from the uniqueness property of solutions of differential equations that $y(x, s)$ is independent of $s$ on $|s| < \sigma_1$. Taking $x = 1$ we obtain

$$
F(z_1(s), z_2(s)) = F(z_1(0), z_2(0)) = F(v_1, v_2), \qquad |s| < \sigma_1. \tag{2.14}
$$

We now differentiate in (2.14) and place $s = 0$. Using (2.10) we get

$$
(\alpha'(0)v_1 + \beta'(0)v_2)F_1(v_1, v_2) + (\gamma'(0)v_1 + \delta'(0)v_2)F_2(v_1, v_2) = 0. \tag{2.15}
$$

From (2.11) we obtain

$$
\begin{aligned}
L\alpha'(0) &= \lambda_1' F_1 + \overset{0}{v}_2(\lambda_1' F_{12} + \lambda_2' F_{22}),\\
L\beta'(0) &= \lambda_1' F_2 - \overset{0}{v}_1(\lambda_1' F_{12} + \lambda_2' F_{22}),\\
L\gamma'(0) &= \lambda_2' F_1 - \overset{0}{v}_2(\lambda_1' F_{11} + \lambda_2' F_{12}),\\
L\delta'(0) &= \lambda_2' F_2 + \overset{0}{v}_1(\lambda_1' F_{11} + \lambda_2' F_{12}),
\end{aligned}
$$

where $L$, $F_i$, $F_{ij}$ are evaluated at $(v_1^0, v_2^0)$, and $\lambda_i$ at $s = 0$. By (2.7) and (2.9) we have for the same evaluations

$$
\lambda_1' F_1 + \lambda_2' F_2 = 0, \quad \overset{0}{v}_1(\lambda_1' F_{11} + \lambda_2' F_{12}) + \overset{0}{v}_2(\lambda_1' F_{12} + \lambda_2' F_{22}) = 0. \tag{2.16}
$$

Hence,

$$
\begin{aligned}
\alpha'(0) + \delta'(0) &= 0,\\
\overset{0}{v}_1\alpha'(0) + \overset{0}{v}_2\beta'(0) &= \lambda_1'(0),\\
\overset{0}{v}_1\gamma'(0) + \overset{0}{v}_2\delta'(0) &= \lambda_2'(0).
\end{aligned}
\tag{2.17}
$$

Now $0 < (\lambda_1'(0))^2 + (\lambda_2'(0))^2$. Placing $a_0 = -\gamma'(0)$, $b_0 = \alpha'(0) = -\delta'(0)$, $c_0 = \beta'(0)$, we conclude that $a_0$, $b_0$, $c_0$ are not all zero, and, using (2.15), that (2.6) holds at $(v_1, v_2)$. The only restriction on $(v_1, v_2)$ was that $v_1^0 v_2 - v_2^0 v_1 \neq 0$. Hence, by continuity, (2.6) holds in $V_0$.

We prove now that constants $a$, $b$, $c$, not all zero, can be chosen so that

$$
(bu_1 + cu_2)F_1(u_1, u_2) - (au_1 + bu_2)F_2(u_1, u_2) = 0, \quad (u_1, u_2) \in V. \tag{2.18}
$$

By our preceding analysis, if $P \in V$ and $V_P$ is an open disk in $V$ having center at $P$, there corresponds to $P$ and $V_P$ a set of constants $a_P$, $b_P$,

$c_P$, not all zero, such that (2.18) holds in $V_P$ with $a_P$, $b_P$, $c_P$ in place of $a$, $b$, $c$. We observe that since the gradient of $F$ is nonvanishing in $V$, if (2.18) holds with two different sets of constants on a domain $D \subset V$, these constants must be proportional. Hence, if we normalize the $a_P$, $b_P$, $c_P$, so that the first one which is not zero is one, then $a_P$, $b_P$, $c_P$ are uniquely determined. We show that the normalized $a_P$, $b_P$, $c_P$ are independent of $P$. Let $P_0$: $(v_1^0, v_2^0)$ be a fixed point in $V$, and let $P_1$: $(v_1^1, v_2^1)$ be an arbitrary second point. Let $\Gamma$: $u_1 = \chi_1(t)$, $u_2 = \chi_2(t)$, $0 \leqq t \leqq 1$, $\chi_i(0) = v_i^0$, $\chi_i(1) = v_i^1$, be a Jordan arc in $V$ with endpoints at $P_0$, $P_1$. Let $E$ be the set of points $t$ on $[0, 1]$ such that $a_P = a_{P_0}$, $b_P = b_{P_0}$, $c_P = c_{P_0}$, where $P$ has coordinates $(\chi_1(t), \chi_2(t))$. $E$ is not void, and has on $[0, 1]$ a least upper bound $t'$, say. Utilizing our remark on the proportionality of constants for a domain $D$, we find first that $t' \in E$, and secondly that $t' = 1$. Thus $a_{P_1} = a_{P_0}$, $b_{P_1} = b_{P_0}$, $c_{P_1} = c_{P_0}$. Taking $a = a_{P_0}$, $b = b_{P_0}$, $c = c_{P_0}$, our conclusion relative to (2.18) follows.

The balance of the proof rests on (2.4) and (2.18). Let $a$, $b$, $c$, be constants, not all zero, for which (2.18) holds. Write $\omega(u_1, u_2) = au_1^2 + 2bu_1u_2 + cu_2^2$. From (2.4) and (2.18) we obtain

$$(2.19) \qquad \omega F_1 = (au_1 + bu_2)\xi(F), \qquad \omega F_2 = (bu_1 + cu_2)\xi(F)$$

for $(u_1, u_2) \in V$. We show first that $\omega \neq 0$ on $V$. Suppose, if possible, that $(v_1, v_2) \in V$ and $\omega(v_1, v_2) = 0$. Since $\xi(F) \neq 0$, it then follows that $av_1 + bv_2 = 0$, $bv_1 + cv_2 = 0$. Since $V$ does not contain the origin, we then have $b^2 - ac = 0$. Not both $a$ and $c$ vanish, and we can assume without loss of generality that $a \neq 0$. In this case we have $\omega = (au_1 + bu_2)^2/a$ in $V$, and accordingly, by (2.19), $(au_1 + bu_2)^2 F_1 = a(au_1 + bu_2)\xi(F)$ in $V$. From this relation we obtain $(au_1 + bu_2)F_1 = a\xi(F)$ in $V$ except possibly on the line $au_1 + bu_2 = 0$. By continuity we then have $\xi(F) = (av_1 + bv_2)F_1/a = 0$ at $(v_1, v_2)$, which is a contradiction. Thus, $\omega \neq 0$ in $V$.

From (2.19) we have

$$(2.20) \qquad dF = (1/2)\xi(F)\omega^{-1}d\omega, \qquad (u_1, u_2) \in V.$$

Let $\eta$ be arbitrary on $(m, M)$. Motivated by (2.20) we define $\phi$ by (2.3). Then $\phi \in C''$ and $\phi' \neq 0$ on $(m, M)$. From (2.20) we get

$$(2.21) \qquad \phi^{-1}(F)d\phi(F) = (1/2)\omega^{-1}d\omega, \qquad (u_1, u_2) \in V.$$

Since $\omega \neq 0$, we can normalize $a$, $b$, $c$ so that at an arbitrary point $(v_1, v_2)$ in $V$ we have $0 < \omega$ and $\phi(F(v_1, v_2)) = \omega^{1/2}(v_1, v_2)$. The $\omega$ of our theorem is then determined. From (2.21) we obtain $\phi(F) = \omega^{1/2}$ in $V$. But $\phi$ has an inverse $\Phi$. Thus, $F = \Phi(\omega^{1/2})$ in $V$. This completes the proof.

3. **Solutions of (1.2).** We now consider briefly the integration of (1.2) when $f$ has the form (1.3) and $Z$, $A$, $C$ satisfy (1.4). We assume that $A\in C^0$, $Z\in C'$, $C\in C'$, and that the equations (1.4) hold on an interval $m<y<M$. We note that the first equation in (1.4) implies that $Z$ vanishes at most once on $(m, M)$. Also, the two equations imply that $Y=Z^3C$ satisfies $Y'=4AY$. Hence, $ZC\neq 0$ or $ZC\equiv 0$ on $(m, M)$. It suffices then to treat two cases: Case I. $Z\neq 0$ on $(m, M)$; Case II. $C\equiv 0$ and $Z=0$ at some one point of $(m, M)$.

*Case* I. For the case $Z\neq 0$, Theorem 2 is directly applicable. For $f$ in the form (1.3), $f(y, 0, 0, 1)\equiv Z(y)$. We then define $\phi$ by (2.3) with $\xi=Z$ and $\eta$ arbitrary on $(m, M)$. We may observe that because of (1.4) $\phi$ can be written

$$(3.1)\quad \phi(y) = Z^{-1}(\eta)Z(y)\exp\left(-\int_\eta^y A(t)\,dt\right), \qquad m<y<M.$$

Denoting by $\Phi$ the inverse of $\phi$, we have $\Phi(t)\in C''$, $\Phi\neq 0$ on $g<t<G$, where $(g, G)$ is the range of $\phi$. Plainly, $0\leqq g$. It remains to determine $\omega$. Let $\omega(u_1,u_2)=au_1^2+2bu_1u_2+cu_2^2$ be an arbitrary homogeneous polynomial of degree 2 which is positive at some point in the plane. Put $\Delta=b^2-ac$, and let $V$ be the (or a) domain determined by $0<\omega$, $g<\omega^{1/2}<G$. Suppose that $w(x)$, $q(x)$ are arbitrary functions satisfying $w\in C'$, $w\neq 0$, $q\in C^0$ on an interval $J$. Let $x_0$ be a point of $J$ and let $y_0$, $y_0'$ be arbitrary initial values, where $m<y_0<M$. Evidently we can find $(v_1, v_2)$ in $V$ such that $\omega_0^{1/2}\equiv\omega^{1/2}(v_1, v_2)=\phi(y_0)$. Furthermore, $v_1'$, $v_2'$ can be determined so that $v_1v_2'-v_2v_1'=w(x_0)$, $\Phi(\omega_0^{1/2})[v_i, v_i']\omega_0^{-1/2}=y_0'$, where $[v_i, v_i']=av_1v_1'+bv_1v_2'+bv_1'v_2+cv_2v_2'$. Denote by $u_1(x)$, $u_2(x)$ the solutions of (1.1) on $J$ with initial values $u_i(x_0)=v_i$, $u_i'(x_0)=v_i'$. Then $u_1(x)$, $u_2(x)$ have Wronskian $w$ and $(u_1, u_2)\in V$ for $x$ in an interval $I$ containing $x_0$. Defining $y$ by $(1/2)\ln\omega=\int_\eta^y Z^{-1}(t)\,dt$, and making use of (1.1), the first equation in (1.4) and the identity

$$(3.2)\qquad [u_i, u_i][u_i', u_i'] = [u_i, u_i']^2 - \Delta(u_1u_2' - u_2u_1')^2,$$

we find that $y=y(x)=\Phi(\omega^{1/2}(u_1(x), u_2(x)))$ satisfies

$$(3.3)\quad y'' = w'(x)w^{-1}(x)y' + q(x)Z(y) + A(y)(y')^2 - \Delta Z(y)\omega^{-2}w^2(x).$$

But

$$(3.4)\ C(y) = C(\eta)Z^3(\eta)Z^{-3}(y)\exp\left(4\int_\eta^y A(t)\,dt\right) = C(\eta)Z^{-1}(\eta)Z(y)\omega^{-2}.$$

Hence, $y$ satisfies (1.2) on $I$ if $\Delta=-C(\eta)Z^{-1}(\eta)$. We have, further, $y(x_0)=\Phi(\omega_0^{1/2})=y_0$, $\quad y'(x_0)=\Phi'(\omega_0^{1/2})[v_i,v_i']\omega_0^{-1/2}=y_0'$. Accordingly, for Case I, if $\Delta = -C(\eta)Z^{-1}(\eta)$, then for arbitrary $w(x)$, $q(x)$ having the properties prescribed above, $y=\Phi(\omega^{1/2}(u_1, u_2))$, together with solutions $u_1(x)$, $u_2(x)$ of (1.1) having Wronskian $w$, and such that $(u_1(x), u_2(x))\in V$ provide the general solution of (1.2).

*Case* II. If $Z$ vanishes at a point of $(m, M)$, then the definition (2.3) is inappropriate. However, if $\eta$ is chosen so that $Z(\eta)\neq 0$, then (3.1) is applicable. Defining $\phi$ by (3.1), and using the first equation in (1.4), we find that

$$\phi'(y) = Z^{-1}(\eta)\exp\left(-\int_\eta^y A(t)\,dt\right), \qquad m < y < M. \tag{3.5}$$

Thus, $\phi\in C''$ and $\phi'\neq 0$ on $(m, M)$. Denoting by $\Phi$ the inverse of $\phi$, we have again $\Phi(t)\in C''$, $\Phi'\neq 0$ on $g<t<G$, where $(g, G)$ is the range of $\phi$. In this case, $g<0<G$. Let $\tilde{\omega}(u_1, u_2)=au_1+bu_2$ be an arbitrary nontrivial linear function. Determine $V$ by $g<\tilde{\omega}<G$. Introducing $w, q, u_1, u_2$ as in the preceding paragraph, we find that $y=\Phi(au_1+bu_2)$ satisfies

$$y'' = w'w^{-1}y' + qZ(y) + A(y)(y')^2. \tag{3.6}$$

Since in the case under consideration we have $C\equiv 0$, we see that $y=\Phi(\tilde{\omega}(u_1, u_2))$, together with appropriate solutions of (1.1), provide the general solution (1.2) for Case II.

We may observe that canonical forms for the solution of (1.2) are

$$y = \Phi([u_1^2 + u_2^2]^{1/2}), \quad y = \Phi([u_1^2 - u_2^2]^{1/2}), \quad y = \Phi(u_1),$$

where $u_1$, $u_2$ are solutions of (1.1).

## References

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