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PURDUE UNIVERSITY

INJECTIVE MODULES UNDER CHANGE OF RINGS1

ERNST SNAPPER

Introduction. Let U and V be rings with unit element and $k \colon U \to V$ a ring epimorphism with kernel I. Every V-module (all modules are left modules) can be regarded as a U-module under k and hence it makes sense to ask when a V-module is U-injective.

For $u \in U$, we denote the left ideal $\{c \mid c \in U, cu = 0\}$ by 0:u. The answer to the above question is simply:

Criterion. A V-module A is U-injective if and only if it satisfies the following two conditions:

- (1) A is V-injective;
- (2) If $u \in I$ and $a \in A$ and (0:u)a = 0, then a = 0. (The cutest way to put (2) is 0:(0:u) = 0 for all $u \in I$.)

We prove the criterion in $\S 1$ and make an application of it to G-modules in $\S 2$. (G stands for a finite group.)

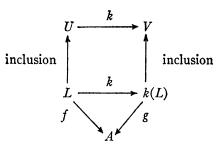
1. Proof of the criterion. Let the V-module A be U-injective. In order to prove condition 1, we select a left ideal M of V and a V-homomorphism $g: M \rightarrow A$. We have to produce an element $a \in A$ such that g(v) = va for all $v \in M$. (See [1, Theorem 3.2, p. 8].) Hereto we consider the left ideal $k^{-1}(M)$ of U and the U-homomorphism $gk: k^{-1}(M) \rightarrow A$. Since A is U-injective, there exists an $a \in A$ such that gk(u) = ua for all $u \in k^{-1}(M)$. Let now $v \in M$. Since k is an epi, v = k(u) for some $u \in k^{-1}(M)$ and, hence, g(v) = gk(u) = ua. The action of U on A is such that ua = k(u)a = va and condition 1 is proved.

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In order to prove condition 2, we consider a left ideal L of U and a U-homomorphism $f: L \rightarrow A$. Again, there exists an $a \in A$ such that f(u) = ua = k(u)a for all $u \in L$. Hence, if $u \in L \cap I$, f(u) = 0. Suppose now that $u \in I$ and $a \in A$ and that (0:u)a = 0. There exists a U-homomorphism $f: (u) \rightarrow A$, where (u) stands for the left ideal generated by u and where f(cu) = ca for all $c \in U$. Since f must be the zero mapping, a = 0 and condition 2 is proved.

We now assume that the V-module A satisfies conditions 1 and 2, and show that A is U-injective. Hereto, we select a left ideal L of U and a U-homomorphism $f\colon L\to A$. We have to show that there exists some $a\in A$ such that f(u)=ua for all $u\in L$. If $u\in L$, obviously $(0\colon u)f(u)=0$ and, hence, condition 2 tells us that $f(L\cap I)=0$. Since $L\cap I$ is the kernel of the restriction of k to L, there exists a U-homomorphism $g\colon k(L)\to A$ such that f=gk. The following diagram will be helpful.



The square and the triangle both commute. Since k is an epi, k(L) is a left ideal of V and g is a V-homomorphism. We conclude from condition 1 that there exists an $a \in A$ such that g(v) = va for all $a \in k(L)$. Let now $u \in L$. Then f(u) = gk(u) = k(u)a = ua. Done.

COROLLARY. If I contains a left nondivisor of zero, the only V-module which is U-injective is the zero module. ($u \in U$ is a left nondivisor of zero if $cu \neq 0$ for all nonzero $c \in U$.)

PROOF. If $u \in I$ is a left nondivisor of zero, 0: u = 0. Hence condition 2 now states that all elements of A are zero. Done.

2. G-modules. Let G be a finite group of order n. The customary ring epimorphism $\mathcal{E}: Z[G] \to Z$, where Z is the ring of the rational integers and Z[G] is the integral group ring of G, is given by $\mathcal{E}(z_1\sigma_1 + \cdots + z_n\sigma_n) = z_1 + \cdots + z_n$; here, $G = \{\sigma_1, \cdots, \sigma_n\}$ and $z_1, \cdots, z_n \in Z$. The kernel of \mathcal{E} is denoted by I and the left ideal $\{u \mid u \in Z[G], uc = 0\}$ for all $c \in I\}$ by 0:I. The "trace" $\sigma_1 + \cdots + \sigma_n$ is designated by S and we recall that $uS = Su = \mathcal{E}(u)S$ for all $u \in Z[G]$. It follows that

the principal ideal (S), generated by S, consists of the integral multiples of S.

PROPOSITION. 0: S = I and 0: I = (S). There exist elements $u \in I$ such that 0: u = (S).

PROOF. The fact that 0:S=I follows from $uS=\mathcal{E}(u)S$. We conclude from it that $(S)\subset 0:I$. For each $\sigma\in G$, $S-n\sigma\in I$ and one checks easily that $0:(S-n\sigma)=(S)$. This also shows that $0:I\subset (S)$ and we are done.

We are now ready to apply the criterion. As customary, G-module or G-injective means Z[G]-module or Z[G]-injective.

THEOREM. Let A be an abelian group on which G acts trivially. Then, A is G-injective if and only if A is divisible and $na \neq 0$ for all nonzero $a \in A$.

PROOF. To say that G acts trivially on A is the same as to say that A is considered as a G-module under the epimorphism $E: Z[G] \rightarrow Z$. Hence, the criterion may be applied. Condition 1 states that A is Z-injective, i.e., divisible. (See Corollary 7.3, p. 93 of [2].) We see from the above proposition that condition 2 now states that, if Sa=0 for some $a \in A$, then a=0. Since Sa=na we are done.

The theorem of this section is not new. It is contained in Rim's general result on G-injectiveness and G-projectiveness, formulated as Proposition 2.3 of [3]. The theorem in question is needed for the author's forthcoming paper on the duality of the cohomology of permutation representations.

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DARTMOUTH COLLEGE