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INJECTIVE MODULES UNDER CHANGE OF RINGS[1]

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Introduction. Let U and V be rings with unit element and $k: U \to V$ a ring epimorphism with kernel I. Every V-module (all modules are left modules) can be regarded as a U-module under k and hence it makes sense to ask when a V-module is U-injective.

For $u \in U$, we denote the left ideal $\{c \mid c \in U, cu = 0\}$ by $0:u$. The answer to the above question is simply:

Criterion. A V-module A is U-injective if and only if it satisfies the following two conditions:

(1) *A is V-injective*;

(2) *If $u \in I$ and $a \in A$ and $(0:u)a = 0$, then $a = 0$.* (The cutest way to put (2) is $0:(0:u) = 0$ for all $u \in I$.)

We prove the criterion in §1 and make an application of it to G-modules in §2. (G stands for a finite group.)

1. **Proof of the criterion.** Let the V-module A be U-injective. In order to prove condition 1, we select a left ideal M of V and a V-homomorphism $g: M \to A$. We have to produce an element $a \in A$ such that $g(v) = va$ for all $v \in M$. (See [**1**, Theorem 3.2, p. 8].) Hereto we consider the left ideal $k^{-1}(M)$ of U and the U-homomorphism $gk: k^{-1}(M) \to A$. Since A is U-injective, there exists an $a \in A$ such that $gk(u) = ua$ for all $u \in k^{-1}(M)$. Let now $v \in M$. Since k is an epi, $v = k(u)$ for some $u \in k^{-1}(M)$ and, hence, $g(v) = gk(u) = ua$. The action of U on A is such that $ua = k(u)a = va$ and condition 1 is proved.

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In order to prove condition 2, we consider a left ideal L of U and a U-homomorphism $f\colon L \to A$. Again, there exists an $a \in A$ such that $f(u) = ua = k(u)a$ for all $u \in L$. *Hence, if* $u \in L \cap I$, $f(u) = 0$. Suppose now that $u \in I$ and $a \in A$ and that $(0:u)a = 0$. There exists a U-homomorphism f: $(u) \to A$, where (u) stands for the left ideal generated by u and where $f(cu) = ca$ for all $c \in U$. Since f must be the zero mapping, $a = 0$ and condition 2 is proved.

We now assume that the V-module A satisfies conditions 1 and 2, and show that A is U-injective. Hereto, we select a left ideal L of U and a U-homomorphism $f\colon L \to A$. We have to show that there exists some $a \in A$ such that $f(u) = ua$ for all $u \in L$. If $u \in L$, obviously $(0:u)f(u) = 0$ and, hence, condition 2 tells us that $f(L \cap I) = 0$. Since $L \cap I$ is the kernel of the restriction of k to L, there exists a U-homomorphism g: $k(L) \to A$ such that $f = gk$. The following diagram will be helpful.

$$\begin{tikzcd}
U \arrow[rr, "k"] & & V \\
L \arrow[u, "\text{inclusion}"] \arrow[rr, "k"] \arrow[dr, "f"'] & & k(L) \arrow[u, "\text{inclusion}"'] \arrow[dl, "g"] \\
& A &
\end{tikzcd}$$

The square and the triangle both commute. Since k is an epi, $k(L)$ is a left ideal of V and g is a V-homomorphism. We conclude from condition 1 that there exists an $a \in A$ such that $g(v) = va$ for all $a \in k(L)$. Let now $u \in L$. Then $f(u) = gk(u) = k(u)a = ua$. Done.

COROLLARY. *If I contains a left nondivisor of zero, the only V-module which is U-injective is the zero module.* ($u \in U$ *is a left nondivisor of zero if* $cu \neq 0$ *for all nonzero* $c \in U$.)

PROOF. If $u \in I$ is a left nondivisor of zero, $0:u = 0$. Hence condition 2 now states that all elements of A are zero. Done.

2. G-**modules.** Let G be a finite group of order n. The customary ring epimorphism $\mathcal{E}\colon Z[G] \to Z$, where Z is the ring of the rational integers and $Z[G]$ is the integral group ring of G, is given by $\mathcal{E}(z_1\sigma_1 + \cdots + z_n\sigma_n) = z_1 + \cdots + z_n$; here, $G = \{\sigma_1, \cdots, \sigma_n\}$ and $z_1, \cdots, z_n \in Z$. The kernel of $\mathcal{E}$ is denoted by I and the left ideal $\{u \mid u \in Z[G], uc = 0$ for all $c \in I\}$ by $0:I$. The "trace" $\sigma_1 + \cdots + \sigma_n$ is designated by S and we recall that $uS = Su = \mathcal{E}(u)S$ for all $u \in Z[G]$. It follows that

the principal ideal (S), generated by S, consists of the integral multiples of S.

PROPOSITION. $0:S=I$ *and* $0:I=(S)$. *There exist elements* $u\in I$ *such that* $0:u=(S)$.

PROOF. The fact that $0:S=I$ follows from $uS=\mathcal{E}(u)S$. We conclude from it that $(S)\subset 0:I$. For each $\sigma\in G$, $S-n\sigma\in I$ and one checks easily that $0:(S-n\sigma)=(S)$. This also shows that $0:I\subset(S)$ and we are done.

We are now ready to apply the criterion. As customary, G-module or G-injective means $Z[G]$-module or $Z[G]$-injective.

THEOREM. *Let A be an abelian group on which G acts trivially. Then, A is G-injective if and only if A is divisible and $na\neq 0$ for all nonzero $a\in A$.*

PROOF. To say that G acts trivially on A is the same as to say that A is considered as a G-module under the epimorphism $\mathcal{E}: Z[G]\rightarrow Z$. Hence, the criterion may be applied. Condition 1 states that A is Z-injective, i.e., divisible. (See Corollary 7.3, p. 93 of [2].) We see from the above proposition that condition 2 now states that, if $Sa=0$ for some $a\in A$, then $a=0$. Since $Sa=na$ we are done.

The theorem of this section is not new. It is contained in Rim's general result on G-injectiveness and G-projectiveness, formulated as Proposition 2.3 of [3]. The theorem in question is needed for the author's forthcoming paper on the duality of the cohomology of permutation representations.

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