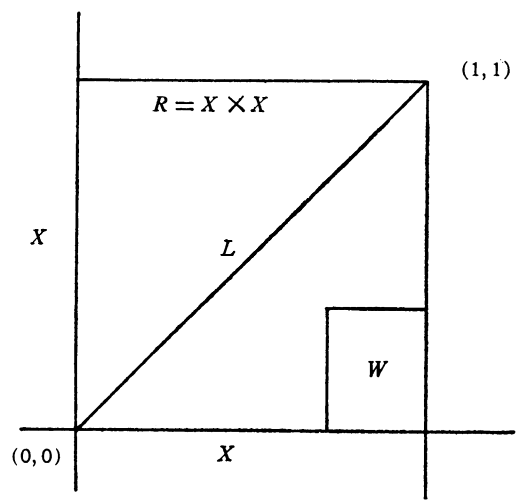
A TRANSLATION OF THE NORMAL MOORE SPACE CONJECTURE[1]

R. H. BING

1. Introduction. The purpose of this note is to translate an unsolved problem in topology into a nontopological setting so that it can be considered by a wider audience. Those interested in the foundations of set theory might find the new formulation more to their liking. A quickening of interest in logic generated by Paul Cohen's result that the Continuum Hypothesis is independent of the axioms of set theory (Zermelo-Fraenkel or Godel-Bernays) together with the Axiom of Choice suggests that this is an opportune time to consider this reformulation.

Let X denote a set of points, R the cartesian product $X\times X$, and L the diagonal of R consisting of all points $(x, x)\in X\times X$. It may be convenient to think of R as a unit square in the Euclidean plane with diagonal L from (0, 0) to (1, 1) as shown in Figure 1. However, we do not insist that the cardinality of X be that of the continuum.



The horizontal projection $h(x, y)=(y, y)$ and the vertical projection $v(x, y)=(x, x)$ sends R onto L. We shall be concerned with sets W as shown in Figure 1 such that $h(W)\cap v(W)=0$. Although W may be as shown, it may be more complicated (such as the set of points

Received by the editors June 5, 1964.

[1] Work on this paper supported by the National Science Foundation under grant GP-2244.

of R with rational abscissas and irrational ordinates, or worse).

Let $f: R-L \rightarrow \{0, 1, 2, \cdots\}$ be a transformation (not necessarily continuous) of $R-L$ into the nonnegative integers. Do the following possible properties of f imply each other?

(a) There is a transformation $F: X \rightarrow \{0, 1, 2, \cdots\}$ such that $\max[F(x), F(y)] > f(x, y)$ for each $(x, y) \in R-L$.

(b) For each subset W of R with $h(W) \cap v(W) = 0$, there is a transformation $F_W: X \rightarrow \{0, 1, 2, \cdots\}$ such that $\max[F_W(x), F_W(y)] > f(x, y)$ for each $(x, y) \in W$.

It is clear that Condition (a) implies Condition (b) but whether or not Condition (b) implies Condition (a) is not obvious. Whether or not there is such an implication is related to the following conjecture.

Normal Moore space conjecture. Each normal Moore space is metrizable.

A Moore space is one which satisfies the first three parts of Axiom 1 of [6]. Hence a Moore space is a regular Hausdorff space S with a sequence of open coverings $H_1, H_2, \cdots$ such that for each point p, $\{\text{Star}(p, H_1), \text{Star}(p, H_2), \cdots\}$ is a countable basis for p. The sequence of open coverings $H_1, H_2, \cdots$ is called a development. Sometimes additional conditions are imposed such as that each H_{i+1} is a refinement of H_i (or even a subcollection of H_i). Although these extra conditions are not essential, we suppose for convenience that H_{i+1} refines H_i.

An alternate way of characterizing a Moore space is to say that it is a regular Hausdorff space such that for each point p of it, there is a countable basis $U(p, 1), U(p, 2), \cdots$ for p such that for each open set V containing a point q, there is an integer $n(q, V)$ such that $q \in U(r, n(q, V))$ implies $U(r, n(q, V)) \subset V$.

Efforts to show that the normal Moore space conjecture is true have been unsuccessful. See for example, [1, 2, 3, 4, 5, 7, 8]. Similarly, efforts to construct counterexamples have been frustrating. As may be noted by the tone of this paper, the author does not believe that the conjecture is true. We suggest where a counterexample may lie and show that if there are counterexamples of certain sorts, there are perhaps simpler ones.

A collection of closed sets is *discrete* if the closed sets are mutually exclusive and any subset of the collection has a closed sum. A space is *collectionwise normal* with respect to a discrete collection $\{K_\alpha\}$ of closed sets if for each element $K_\alpha \in \{K_\alpha\}$ there is an open set O_α containing K_α such that all the sets O_α are mutually exclusive.

Counterexample of Type D. It is known [1] that if there is a counterexample S to the normal Moore space conjecture, then S contains a discrete collection of closed sets such that S is not collectionwise normal with respect to the collection. We say that S is a *counterexample of Type* D if it has the additional property that it contains a discrete collection of points with respect to which it is not collectionwise normal.

Question. Does the existence of a counterexample to the normal Moore space conjecture imply the existence of one of Type D? By using a result of Worrell [7], Traylor [2] has shown that there is a counterexample of Type D if there is a locally compact counterexample. Theorem 3 shows that each separable counterexample is of Type D.

2. The nonimplication gives a counterexample.

THEOREM 1. *If there is an* X, R, L, $f(x, y)$ *satisfying Condition* (b) *but not Condition* (a), *then there is a counterexample of Type* D.

PROOF. The points of the counterexample are the points of R. Each point of $R-L$ is isolated. For each point $(x, x)\in L$ and each integer n,

$$N((x, x), n) = \{(x, x)\} \cup \{(y, z) \in R \mid f(y, z) > n \text{ and } x \text{ is either } y \text{ or } z\}.$$

Note that $N((x, x), n)$ lies on the sum of the horizontal and vertical segments in R through (x, x). Elements of the open covering H_i of the development $H_1, H_2, \cdots$ are either the one point neighborhoods containing only one point of $R-L$ or neighborhoods of the sort $N((x, x), i)$.

To see that the space is normal, consider two mutually exclusive closed sets A, B in R. Let W_1 be the set of all points (x, y) of R such that $(x, x)\in A$ and $(y, y)\in B$. Also, let $W_2=\{(x, y) \mid (x, x)\in B, (y, y)\in A\}$. Note that $h(W_1)\cap v(W_1)=0=h(W_2)\cap v(W_2)$. Let F_{W_1}, F_{W_2} be the functions promised by Condition (b). Let D_A be the open set consisting of the sum of all $N((a, a), n)$'s with $(a, a)\in A$ and $n=F_{W_1}(a)+F_{W_2}(a)$. Also, let D_B be the sum of all $N((b, b), m)$'s such that $(b, b)\in B$ and $m=F_{W_1}(b)+F_{W_2}(b)$. The only possible point of intersection of $N((a, a), n)$ and $N((b, b), m)$ is either the point (a, b) or the point (b, a). However, $(a, b)\in W_1$ and since $\max[m, n] \geqq \max[F_{W_1}(a), F_{W_1}(b)]>f(a, b)$, (a, b) does not belong to both $N((a, a), n)$ and $N((b, b), m)$. Similarly, we find that $(b, a)\in W_2$ does not belong to both. Then $O_A=(D_A-B)\cup A$ and $O_B=(D_B-A)\cup B$ are mutually exclusive open subsets of R containing A and B respectively.

Finally we show that the topology we have given R makes it a counterexample of Type D unless $f(x, y)$ satisfies Condition (a). Assume R is collectionwise normal with respect to the collection of points in L and let $\{U_x\}$ be a collection of mutually exclusive open sets covering L such that U_x is the only element of $\{U_x\}$ containing $(x, x) \in L$. Let $F(x)$ be a positive integer such that $N((x, x), F(x)) \subset U_x$. Then $F(x)$ shows that Condition (a) is satisfied. If $(x, y) \in R-L$, then $(x, y) \notin N((x, x), F(x)) \cap N((y, y), F(y))$ and (x, y) fails to belong to one of them, say $N((x, x), F(x))$. But then $F(x) \geqq f(x, y)$ and $\max[F(x), F(y)] \geqq f(x, y)$.

The following theorem shows that if Condition (b) fails to imply Condition (a), then X must be uncountable.

THEOREM 2. *Each normal Moore space is collectionwise normal with respect to countable discrete collections of closed sets.*

PROOF. Let $A_1, A_2, \cdots$ be a countable discrete collection of closed sets in a normal Moore space S. For each integer i, let D_i and D_i' be mutually exclusive open sets containing A_i and $\bigcup_{j \neq i} A_j$ respectively. Then $D_1, D_1' \cap D_2, D_1' \cap D_2' \cap D_3, \cdots$ are mutually exclusive collections of open sets showing that S is collectionwise normal with respect to $A_1, A_2, \cdots$.

Question. Let $\aleph'$ be the least cardinal such that some normal Moore space contains a discrete collection X of closed sets such that S is not collectionwise normal with respect to X and cardinality of X is $\aleph'$. What is the size of $\aleph'$? The proof of Theorem 2 shows that if $\aleph'' < \aleph'$, then $\aleph'$ is not the limit of $\aleph''$ cardinals all less than $\aleph'$.

THEOREM 3. *Each separable counterexample to the normal Moore space conjecture is of Type* D.

PROOF. It follows from [1] that each counterexample contains a discrete collection of closed sets with respect to which the counterexample is not collectionwise normal. Selecting one point from each element of this collection, one finds from Theorem 2 that the counterexample contains an uncountable discrete collection of points. The space is not collectionwise normal with respect to this collection since no separable space is collectionwise normal with respect to any uncountable collection.

3. A converse of Theorem 1.

THEOREM 4. *If S is a counterexample of Type* D *and X is a discrete collection of points with respect to which S is not collectionwise normal, then there is a function $f(x, y)$ showing that Condition* (b) *does not imply Condition* (a).

PROOF. Assume S is a counterexample of Type D, X is a discrete collection of points with respect to which S is not collectionwise normal, and $H_1, H_2, \cdots$ is a development of S. For each $(x, y) \in R-L$, let $f(x, y)=0$ if $\text{Star}(x, H_1) \cap \text{Star}(y, H_1)=0$ and otherwise let $f(x, y)=n$ where n is the largest integer for which $\text{Star}(x, H_n) \cap \text{Star}(y, H_n) \neq 0$. Note that $\text{Star}(x, H_{f(x,y)+1}) \cap \text{Star}(y, H_{f(x,y)+1})=0$. We show that $f(x, y)$ satisfies Condition (b) but not Condition (a).

Let W be a subset of R such that $h(W) \cap v(W) = 0$. Let $A = \{x \in X \mid (x, y) \in W\}$ and $B=X-A$. Since S is normal, there are mutually exclusive open sets D_A, D_B in S containing A and B respectively.

Let F_W be a transformation of X into the positive integers such that $\text{Star}(x, H_{F_W(x)})$ lies in one of D_A, D_B. For each point $(x, y) \in W$, $h(x, y)$, $v(x, y)$ belong to different ones of A, B and $N(x, H_{F_W(x)})$, $N(y, H_{F_W(y)})$ lie in different ones of D_A, D_B. Hence $N(x, H_{F_W(x)}) \cap N)y, H_{F_W(y)})=0$ and for $m=\max[F_W(x), F_W(y)]$, $N(x, H_m) \cap N(y, H_m)=0$. Hence $f(x, y)<m$ and $\max[F_W(x), F_W(y)]>f(x, y)$. Hence Condition (b) is satisfied.

We show that Condition (a) is not satisfied by showing that if it were, then S would be collectionwise normal with respect to X. Assume there is an F showing that Condition (a) is satisfied. Let X_i be the set of all points $x \in X$ such that $F(x)=i$. It follows from Theorem 2 that there is a collection of mutually exclusive open sets $D_1, D_2, \cdots$ in S containing $X_1, X_2, \cdots$ respectively. For each point $x \in X_i$, let $U_x=D_i \cap \text{Star}(x, H_{F(x)})$. We show that the assumption that there is an F leads to the contradiction that $\{U_x\}$ is a collection of mutually exclusive open sets showing that S is collectionwise normal with respect to X.

Assume $U_{x_1} \cap U_{x_2} \neq 0$ for some pair of different points x_1, x_2 of X. Then x_1, x_2 belong to the same X_i since $D_i \cap D_j=0$ if $i \neq j$. Suppose $x_1, x_2 \in X_{n_1}$. Then $F(x_1)=F(x_2)=n_0 \geq f(x_1, x_2)+1$. Since

$$U_{x_i} \subset \text{Star}(x_i, H_{F(x_i)}) \subset \text{Star}(x_i, H_{f(x_1,x_2)+1}),$$

$$U_{x_1} \cap U_{x_2} \subset \text{Star}(x_1, H_{f(x_1,x_2)+1}) \cap \text{Star}(x_2, H_{f(x_1,x_2)+1}) = 0.$$

The assumption that Condition (a) is satisfied led to the contradiction that $U_{x_1} \cap U_{x_2} \neq 0 = U_{x_1} \cap U_{x_2}$.

We can state Theorems 1 and 4 together.

THEOREM 5. *A necessary and sufficient condition that there be a counterexample of Type* D *is that there be an* X, R, L, $f(x, y)$ *satisfying Condition* (b) *but not Condition* (a).

4. **Changing counterexamples.** Suppose one took a counterexample S' of Type D, used Theorem 4 to get an X, R, L and $f(x, y)$ satis-

fying Condition (b) but not Condition (a) and finally used the X, R, L, $f(x, y)$ to get a counterexample S'' of Type D as shown in Theorem 1. The resulting S'' might be simpler that S'. How is S'' related to S'?

One finds that S'' is obtained from S' by isolating points of $S'-X$, throwing certain of these isolated points away, combining others, and splitting others. Let us do it in slow motion.

In the following we suppose that S' is a counterexample of Type D and X is a discrete set of points with respect to which S' is not collectionwise normal. A *partition* $[X_\alpha, X_\alpha']$ of X is a separation of X into two mutually exclusive subsets.

EXAMPLE 1. Let X be an uncountable set, F be the set of all functions $f: X \to \{0, 1, 2, \cdots\}$, G is the set of bounded elements of F, and G' be a subset of G such that

(c) For each $f \in F$, there is a $g \in G'$ such that $g > f$ on two points of X.

(d) For each partition $[X_\alpha, X_\alpha']$ of X, there is an $f_\alpha \in F$ such that for each $g \in G'$, either $f_\alpha > g$ for each value in X_α or for each value in X_α'.

Our description of a G' satisfying the conditions of Example 1 is dependent on the assumption that there is a counterexample S' of Type D with a discrete set X with respect to which S' is not collectionwise normal. Before describing a G' we point out how the existence of such a G' (no matter how obtained) insures the existence of a counterexample S_1 of Type D.

Let S_1 be the topological space whose points are the elements of $X \cup G'$. Each element of G' is isolated. For each $x \in X$ and each positive integer n, $N(x, n) = \{x\} \cup \{g \in G' \mid g(x) \geqq n\}$ is a neighborhood in the nth element of the development of S_1.

No matter how the set G' is obtained, the resulting S_1 is a counterexample of Type D. Condition (c) implies that S_1 is not collectionwise normal with respect to X and Condition (d) implies that S_1 is normal.

If S' is a normal Moore space with a discrete collection X of points with respect to which S' is not collectionwise normal, the collection G' may be obtained as follows. For each $p \in S'-X$, associate the function $g_p \in G$ such that $g_p(x)$ = largest integer n such that $p \in \operatorname{Star}(x, H_n)$. Then G' is the set of elements of G that are associated with some point of $S'-X$.

Note that in changing from S' to S_1, we isolated the points of $S'-X$ and combined the ones of these isolated points that were associated with the same element of G'.

We note that S_1 does not have more points than S'. We also note that if $C(Z)$ denotes the cardinality of set Z, then $C(S_1) \leqq 2^{C(X)}$.

EXAMPLE 2. This example is the same as Example 1 with the following additional restriction.

(e) Each $g \in G'$ is positive on precisely two values of X.

The G' of Example 2 is obtained from the G' of Example 1 by first throwing away all the elements of G' of Example 1 which are positive on no more than one element of X and then for each other $g' \in G'$ of Example 1, replace this g' with as many elements of G' of Example 2 as there are pairs of points of X on which g' is positive.

If S_2 represents the resulting counterexample of Type D, we obtained S_2 from S_1 by throwing away certain isolated points of S_1, splitting others, and combining certain of the split pieces. Note that after the discarding, splitting, and combining, we have not increased the cardinality since $C(S_2) = C(X) \leqq C(S_1)$.

EXAMPLE 3. This example is the same as Example 2 with the following additional restriction.

(f) If $g \in G'$, $g(x_1) > 0$, $g(x_2) > 0$, then $g(x_1) = g(x_2)$.

If $g' \in G'$ of Example 2 and x, y are different points of X on which g' is positive, then the element g'' of G' of Example 3 corresponding to g' satisfies $g''(x) = g''(y) = \min[g'(x), g'(y)]$ and $g'' = 0$ on $X - (\{x\} \cup \{y\})$.

At first glance it might appear that in reducing the elements of G' we lost Condition (c) but it is not lost. Consider an $f \in F$. To find an element of the new G' that dominates it on two points, we let $X_i = f^{-1}(i)$ and note from the proof of Theorem 4 that there is an integer j such that S_2 is not collectionwise normal with respect to X_j. For some two points x, y of X_j, $\operatorname{Star}(x, H_{j+1}) \cap \operatorname{Star}(y, H_{j+1}) \neq 0$ in S_2. Hence there is a $g \in G'$ of Example 2 such that $g(x) > j$ and $g(y) > j$. The associated element of G' of Example 3 dominates f on x and on y.

If S_3 is the space resulting from Example 3, S_3 is obtained from S_2 by removing certain points from certain neighborhoods and then combining certain of these removed points. When the topology of a space is altered by making the neighborhoods smaller, there is a possibility that a non-collectionwise normal space may become collectionwise normal. However, the proof of Theorem 4 shows that this did not happen.

The function $f(x, y)$ on $R - L$ described in the introduction may be obtained from the G' of Example 3 by letting $f(x, y)$ be the maximal value that any element of G' takes on both x and y. The resulting $f(x, y)$ will satisfy Condition (b) but not Condition (a) if G' satisfies Conditions (c), (d), (e), (f).

The following question is a backhanded version of one that we have asked earlier.

Question. Suppose X is an uncountable set, F is the set of all maps of X into $\{0, 1, 2, \cdots\}$, G is the set of elements of F that are positive on precisely two values of X. Suppose G' is a subcollection of G such that the following holds.

(g) For each $f \in F$, there is a $g \in G'$ such that $g > f$ on the two values of X on which g is positive.

Does Condition (g) imply that G' would necessarily have the following property?

(h) For some partition $[X_\alpha, X_\alpha']$ of X and each $f_\beta \in F$ there is a $g \in G'$ such that $g > f$ for one value of X_α and also for one value of X_α'.

References

1. R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. **8** (1951), 653–663.

2. B. Fitzpatrick, *Metrizability in normal Moore spaces*, Pacific J. Math. (to appear).

3. B. Fitzpatrick and D. R. Traylor, *Two theorems on metrizability of Moore spaces*, Pacific J. Math. (to appear).

4. R. W. Heath, *Screenability, pointwise paracompactness and metrization of Moore spaces*, Canad. J. Math. **16** (1964), 763–770.

5. F. B. Jones, *Concerning normal and completely normal spaces*, Bull. Amer. Math. Soc. **43** (1937), 671–677.

6. R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloq. Publ. Vol. 13, Amer. Math. Soc., Providence, R. I., 1962.

7. J. M. Worrell, *Concerning upper semi-continuous collections of mutually exclusive closed and compact point sets*, Abstract 590-44, Notices Amer. Math. Soc. **9** (1962), 204.

8. J. N. Younglove, *Concerning metric subspaces of nonmetric spaces*, Fund. Math. **48** (1959), 15–25.

THE UNIVERSITY OF WISCONSIN