A NOTE ON SEMIGROUPS OF OPERATORS ON A LOCALLY CONVEX SPACE

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1. The object of this note is to generalize certain results in the theory of semigroups of continuous linear operators on a Banach space to the case where, instead of a Banach space, we consider a locally convex space. A family $\{T(\xi)\}_{\xi>0}$ of linear operators on a vector space is called a semigroup if $T(\xi+\eta)=T(\xi)\circ T(\eta)$, ξ, $\eta\in(0, \infty)$. E. Hille [4] and N. Dunford [2] have proved that if $\{T(\xi)\}_{\xi>0}$ is a semigroup of bounded linear operators on a Banach space E such that for every $x\in E$, $\xi\rightarrow T(\xi)x$ is a measurable function from $(0, \infty)$ into E and such that $\|T(\xi)\|_{\delta\leq\xi\leq 1/\delta}$ is bounded for every $\delta>0(\delta<1)$, then $\xi\rightarrow T(\xi)x$ is a continuous function from $(0, \infty)$ into E for $x\in E$. Proposition 2 is an analogue of this result while Propositions 3 and 4 are analogues of results due to R. S. Phillips [5] and P. Lax, respectively.

2. DEFINITION 1. Let E be a locally convex space, S a set and $\mathfrak{M}$ a σ-ring of "measurable" subsets of S. A function $x: S\rightarrow E$ $(\xi\rightarrow x(\xi))$ is called measurable if it is the limit, almost everywhere, of a sequence of countably valued functions. (A function is said to be countably valued if its range is countable and it takes each value different from zero on a measurable set.)

REMARK. If $x(\xi)$ is a measurable E-valued function and p is any continuous seminorm on E, then $p(x(\xi))$ is a real-valued measurable function on S.

PROPOSITION 1. *Let $S=(0, \infty)$ with $\mathfrak{M}$ the σ-ring of Lebesgue measurable subsets of $(0, \infty)$ and $x(\sigma)$ be a measurable function from $(0, \infty)$ into E. Then for any continuous seminorm q on E, there exists a sequence $\{x_n(\sigma)\}_{n=1,2,\dots}$ of countably valued functions such that $q(x(\sigma)-x_n(\sigma))\rightarrow 0$ as $n\rightarrow\infty$ uniformly for σ outside a set of measure zero.*

We shall prove the proposition under a weaker hypothesis, viz., that $x(\sigma)$ is weakly measurable (i.e., for every continuous linear functional x' on E $\langle x(\sigma), x'\rangle$ is a measurable function of σ on $(0, \infty)$) and that $x(\sigma)$ is almost separably valued (i.e., $x(\sigma)$ belongs to a separable subspace of E almost everywhere).

We may suppose that the range of $x(\sigma)$ is contained in a separable subspace L of E and find a sequence $\{x_n(\sigma)\}_{n\geq 1}$ of countably valued

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functions such that $q(x_n(\sigma)-x(\sigma))$ tends to zero as n tends to infinity uniformly for σ in $(0, \infty)$. Let $\tilde{L}$ denote the quotient space of L, on which q defines a norm $\tilde{q}$. The completion $\tilde{L}^\frown$ of $\tilde{L}$ is a separable Banach space. Hence its conjugate space $(\tilde{L}^\frown)'$ contains a sequence $\{y_n^*\}_{n=1,2,\cdots}$ such that

$$\tilde{q}(\tilde{x}) = \sup_n |\langle \tilde{x}, y_n^*\rangle| \quad \text{for } \tilde{x} \in \tilde{L}^\frown.$$

Now $x \to y_n^*(\tilde{x})$, where $\tilde{x}$ is the image of x by the quotient map $L \to \tilde{L}$, is a continuous linear functional on L. By the Hanh-Banach theorem, there exists a continuous linear functional x_n' on E such that

$$\langle \tilde{x}, y_n^*\rangle = \langle x, x_n'\rangle \quad \text{for } x \in L \text{ and } n = 1, 2, \cdots.$$

Now $q(x(\sigma)) = \tilde{q}[(x(\sigma))^\sim] = \sup_n |\langle x(\sigma), x_n'\rangle|$ is measurable as $\langle x(\sigma), x_n'\rangle$ is measurable for $n=1, 2, \cdots$. Similarly, for any $a \in L$, $q(x(\sigma)-a)$ is a measurable function. Let $A = \{\sigma | q(x(\sigma)) > 0\}$. If $\{a_i\}$ is a sequence dense in L and $A_i = A \cap \{\sigma | q(x(\sigma)-a_i)\} < 1/n$ then $A = \bigcup_{i=1}^\infty A_i$. Let $B_1 = A_1$, $B_j = A_j - \bigcup_{i \leq j-1} B_i$ $(j > 1)$. The B_i are disjoint measurable and $\bigcup B_i = \bigcup A_i = A$.

Let

$$\begin{aligned} x_n(\sigma) &= a_i \quad \text{for } \sigma \in B_i, \\ &= 0 \quad \text{for } \sigma \notin A. \end{aligned}$$

Then $q(x(\sigma)-x_n(\sigma)) < 1/n$ for all σ so that $q(x(\sigma)-x_n(\sigma)) \to 0$ as $n \to \infty$ uniformly for σ in $(0, \infty)$.

DEFINITION 2. A semigroup of operators on a locally convex space E is said to be measurable if, for $x \in E$, the function $\xi \to T(\xi)x$ is measurable from $(0, \infty)$ to E.

PROPOSITION 2. *If $\{T(\xi)\}_{\xi>0}$ is a measurable semigroup of continuous linear operators in a locally convex space E such that, for every $[\alpha, \beta] \subset (0, \infty)$, $\{T(\xi)\}_{\alpha \leq \xi \leq \beta}$ is an equicontinuous family of operators, then $\xi \to T(\xi)x$ is continuous for every $x \in E$.*

PROOF. We have to show that for $\xi \in (0, \infty)$ and $x \in E$,

$$T(\xi \pm \eta)x - T(\xi)x \to 0 \quad \text{in } E \text{ as } \eta \to 0. \tag{1}$$

Let $0 < \alpha < \beta < \xi$. As $\{T(\tau)\}_{\tau \in [\alpha,\beta]}$ is an equicontinuous set, given any continuous seminorm p on E, there exists a continuous seminorm q on E and $k > 0$ such that

$$p(T(\tau)y) \leqq kq(y) \quad \text{for } \alpha \leqq \tau \leqq \beta \text{ and } y \in E. \tag{2}$$

Let $\eta_0 > 0$ be such that $\alpha - \eta > 0$ and $\beta < \xi - \eta$ for $0 < \eta < \eta_0$. By (2), we have

$$p(T(\xi \pm \eta) - T(\xi)x) = p(T(\tau)[T(\xi \pm \eta - \tau)x - T(\xi - \tau)x]) \tag{3}$$
$$\leqq kq(T(\xi \pm \eta - \tau)x - T(\xi - \tau)x).$$

As $\{T(\xi \pm \eta - \tau)\}_{\tau \in [\alpha,\beta]}$ and $\{T(\xi - \tau)\}_{\tau \in [\alpha,\beta]}$ are equicontinuous sets, $\{T(\xi \pm \eta - \tau)x\}_{\tau \in [\alpha,\beta]}$ and $\{T(\xi - \tau)x\}_{\tau \in [\alpha,\beta]}$ are bounded subsets of E so that $q(T(\xi \pm \eta - \tau)x - T(\xi - \tau)x)$ is a bounded measurable function of τ in $[\alpha, \beta]$. Integrating (3) with respect to τ from α to β we get

$$(\beta - \alpha)p(T(\xi \pm \eta)x - T(\xi)x)$$
$$\leqq k \int_\alpha^\beta q(T(\xi \pm \eta - \tau)x - T(\xi - \tau)x)\, d\tau. \tag{4}$$

The integral on the right-hand side tends to zero with η if we show that, for given $\epsilon > 0$, there exists a continuous E-valued function $f_\epsilon(\tau)$ such that

$$\int_{\alpha - \eta_0}^{\beta + \eta_0} q(T(\xi - \tau)x - f_\epsilon(\tau))\, d\tau < \epsilon. \tag{5}$$

For then

$$\int_\alpha^\beta q(T(\xi \pm \eta - \tau)x - T(\xi - \tau)x)\, d\tau$$
$$\leqq \int_\alpha^\beta q(T(\xi \pm \eta - \tau)x - f_\epsilon(\tau \mp \eta))\, d\tau + \int_\alpha^\beta q(f_\epsilon(\tau \mp \eta) - f_\epsilon(\tau))\, d\tau$$
$$+ \int_\alpha^\beta q(f_\epsilon(\tau) - T(\xi - \tau)x)\, d\tau$$

and the first and the third integral are each majorised by ϵ and the second integral tends to zero with η since $f_\epsilon(\tau)$ is continuous. Now $h(\tau) = T(\xi - \tau)x$ being measurable from proposition (1), it follows that, given $\epsilon > 0$, there exists a countably valued function $x(\tau)$ such that $\int_{\alpha-\eta_0}^{\beta+\eta_0} q(h(\tau) - x(\tau))\, d\tau < \epsilon/3$. Let $x(\tau) = \sum_{i=1}^\infty a_i \chi_{A_i}$, where $a_i \in E$ and χ_{A_i} are characteristic functions of disjoint measurable sets A_i contained in $[\alpha - \eta_0, \beta + \eta_0]$. Then we can choose m such that for $y(\tau) = \sum_{i=1}^m a_i \chi_{A_i}$, $\int_{\alpha-\eta_0}^{\beta+\eta_0} q(x(\tau) - y(\tau))\, d\tau < \epsilon/3$. Let $q(y(\tau)) < M$. Given $\delta > 0$ we can find compact sets $K_i \subset A_i$ and open sets $O_i \supset A_i$ $(i = 1, 2, \cdots, m)$ such that $\sum_{i=1}^m m(O_i - K_i) < \delta$ (m being Lebesgue measure). For every i there exists a continuous function $g_i(\tau)$, $0 \leqq g_i(\tau) \leqq 1$, such that $g_i(\tau)$ equals 1 on K_i and 0 outside O_i. If $g(\tau) = \sum_{i=1}^m a_i g_i(\tau)$, then $q(g(\tau)) < mM$ and $g(\tau) = y(\tau)$ for $\tau \notin \bigcup_{i=1}^m (O_i - K_i)$ so that $\int_{\alpha-\eta_0}^{\beta+\eta_0} q(y(\tau) - g(\tau))\, d\tau < 2mM\delta < \epsilon/3$ for $\delta < \epsilon/6mM$. Taking $f_\epsilon(\tau) = g(\tau)$, (5) is satisfied.

REMARK 1. Proposition 2 remains true if instead of the hypothesis that $\xi \rightarrow T(\xi)x$ are E-valued measurable functions for $x \in E$, we have the weaker hypothesis that these functions are weakly measurable and almost separably valued. For, from the proof of Proposition 1, given $\epsilon > 0$, there exists a countably valued function $x(\tau)$ such that $\int_{\alpha-\eta_0}^{\beta+\eta_0} q[h(\tau)x - x(\tau)]\, d\tau < \epsilon/3$ and the proof is completed by the same method as above.

REMARK 2. The converse of Proposition 2, viz., the statement that "If $\{T(\xi)\}_{\xi>0}$ is a semigroup such that $\xi \rightarrow T(\xi)x$ are continuous functions for $x \in E$, then $\{T(\xi)\}_{\xi>0}$ is a measurable semi-group such that for every $[\alpha, \beta] \subset (0, \infty)$, $\{T(\xi)\}_{\xi \in [\alpha,\beta]}$ is an equicontinuous set of maps of E into E" is true if E is barrelled (tonnelé) or is the strong dual of a metrisable locally convex space. For, if $\xi \rightarrow T(\xi)x$ is continuous, then it is measurable and maps the compact set $[\alpha, \beta] \subset (0, \infty)$ onto a compact subset of E. Thus $\{T(\xi)\}_{\alpha \leq \xi \leq \beta}$ is compact in $\mathfrak{L}_s(E, E)$, the space of linear continuous maps of E into E furnished with the topology of simple convergence, and therefore is a closed bounded subset of $\mathfrak{L}_s(E, E)$. If E is barrelled, this implies that $\{T(\xi)\}_{\xi \in [\alpha,\beta]}$ is equicontinuous. If E is the strong dual of a metrisable space, let $(\xi_n)_{n=1,2,\cdots}$ denote the sequence of rationals in $[\alpha, \beta]$. Then $\{T(\xi_n)\}_{n=1,2,\cdots}$ being a subset of $\{T(\xi)\}_{\alpha \leq \xi \leq \beta}$ is bounded in $\mathfrak{L}_s(E, E)$ and is therefore equicontinuous [3, Proposition 1, p. 62]. Hence its closure in $\mathfrak{L}_s(E, E)$ is equicontinuous. Now, for any $\sigma \in [\alpha, \beta]$, there exists a subsequence (ξ_{n_k}) of (ξ_n) such that $\xi_{n_k} \rightarrow \sigma$ as $k \rightarrow \infty$, so that $T(\sigma)x = \lim_{k\to\infty} T(\xi_{n_k})x$ for every $x \in E$. Thus $\{T(\xi)\}_{\alpha \leq \xi \leq \beta}$ is the closure of $\{T(\xi_n)\}_{n=1,2,\cdots}$ in $\mathfrak{L}_s(E, E)$ which is equicontinuous.

PROPOSITION 3. *Let $\{T(\xi)\}_{\xi>0}$ be a measurable semigroup of continuous linear operators in a Fréchet space E. Then for any $[\alpha, \beta] \subset (0, \infty)$,*

$$\{T(\xi)\}_{\alpha \leq \xi \leq \beta}$$

is an equicontinuous set of maps of E into E.

PROOF. Let $\{p_n\}$ be a sequence or seminorms on E defining the topology of E. Since E is Fréchet, it is sufficient to prove that $\{T(\xi)x\}_{\xi \in [\alpha,\beta]}$ is bounded in E for any fixed $x \in E$. If this is not true, there exists an $x \in E$ and an integer i_0 and a sequence $\{\xi_n\}_{n=1,2,\cdots}$ tending to a real number γ where $\xi_n, \gamma \in [\alpha, \beta]$, such that

$$p_{i_0}(T(\xi_n)x) \geqq n \quad \text{for all } n.$$

Let $\{F_i\}_{i=1,2,\cdots}$ be a sequence of measurable subsets of $(0, \gamma]$ such that

(i) $F_{i+1} \subset F_i$,

(ii) the measure $m(F_i)$ of $F_i > \gamma/2 + \gamma/3i$,

(iii) the measurable function $p_i(T(\xi)x)$ is bounded on F_i by M_i, say.

Clearly such a sequence exists and $F = \bigcap_{i=1}^{\infty} F_i$ is measurable with $m(F) \geqq \gamma/2$ and $p_i\{T(\xi)x\} \leqq M_i$ for $\xi \in F$, and $i = 1, 2, \cdots$, i.e., $\{T(\xi)x\}_{\xi \in F}$ is a bounded subset of E.

The sets

$$A_n = \{\xi_n - \eta \mid \eta \in F \cap (0, \xi_n)\}, \qquad n = 1, 2, \cdots$$

are measurable and

$$m(A_n) \geqq \frac{\gamma}{4} \quad \text{for } n \geqq N, \text{ say.} \tag{1}$$

For $\eta \in F \cap (0, \xi_n)$,

$$n = p_{i_0}[T(\xi_n)x] \leqq p_{i_0}[T(\xi_n - \eta)T(\eta)x].$$

Let $\sigma \in A_n$ be arbitrary. Then

$$\sigma = \xi_n - \eta \quad \text{for some } \eta \in F \cap (0, \xi_n),$$

so that

$$p_{i_0}[T(\sigma)T(\eta)x] \geqq n. \tag{2}$$

Let $A = \limsup_{n \to \infty} A_n$. Then $m(A) \geqq \gamma/4$ by (1). For $\sigma_0 \in A$, p_{i_0} is unbounded on $T(\sigma_0)[\{T(\eta)x\}_{\eta \in F}]$ by (2), i.e., $T(\sigma_0)[\{T(\eta)x\}_{\eta \in F}]$ is not a bounded subset of E. This is a contradiction since $T(\sigma_0)$ is a continuous linear map of E into E and $\{T(\eta)x\}_{\eta \in F}$ is a bounded set in E.

Combining Propositions 2 and 3 we have the

THEOREM. *Every measurable semigroup of continuous linear operators on a Frêchet space is a continuous function from* $(0, \infty)$ *into the space of continuous linear maps of E into E furnished with the topology of simple convergence.*

3. PROPOSITION 4. *Let* $\{T(\xi)\}_{\xi > 0}$ *be a measurable semigroup of continuous linear operators on a locally convex space E such that* $\{T(\xi)\}_{\alpha \leqq \xi \leqq \beta}$ *is an equicontinuous family for every* $[\alpha, \beta] \subset (0, \infty)$ *and such that for some* $\xi_0 > 0$, $T(\xi_0)$ *is a compact operator on E. Then, for any* $a > \xi_0$, $T(a)$ *is a compact operator and* $\xi \to T(\xi)$ *is a continuous mapping of* (a, ∞) *into* $\mathfrak{L}_{\mathfrak{S}}(E, E)$ *where* $\mathfrak{L}_{\mathfrak{S}}(E, E)$ *is the space of continuous linear maps of E into E furnished with the topology of uniform convergence on bounded subsets of E.*

PROOF. Let $b > a > \xi_0$ and let B be a bounded subset of E, V a neighbourhood of 0 in E and

$$W(B, V) = \{u \in \mathfrak{L}_{\mathfrak{S}}(E, E) \mid u(B) \subset V\}.$$

We shall show that there exists $\delta > 0$ such that

$$T(\xi + \eta) - T(\xi) \in W(B, V)$$

for $|\eta| < \delta$, $a < \xi < b$, $a < \xi + \eta < b$.

As the sets $\{W(B, V)\}$, where B is a bounded subset of E and V is a neighbourhood of 0 in E, form a fundamental system of neighbourhoods of zero in $\mathfrak{L}_{\mathfrak{S}}(E, E)$, this will prove that $T(\xi+\eta) - T(\xi) \to 0$ in $\mathfrak{L}_{\mathfrak{S}}(E, E)$ as $\eta \to 0$. Let V_1 be a neighbourhood of 0 in E such that

$$V_1 + V_1 + V_1 \subset V. \tag{1}$$

Let V_2 be a neighbourhood of 0 in E such that

$$V_2 \subset V_1 \tag{2}$$

and

$$T(\xi - \xi_0)V_2 \subset V_1 \quad \text{for } a \leqq \xi \leqq b. \tag{3}$$

This is possible since $\{T(\xi - \xi_0)\}_{a \leqq \xi \leqq b}$ is an equicontinuous family of maps of E into E.

Now, $T(\xi_0)$ being a compact operator, there exists a neighbourhood U of 0 such that $T(\xi_0)U$ is a relatively compact subset of E. The bounded subset B of E is absorbed by U, so that $T(\xi_0)B$ is also relatively compact. Given a neighbourhood V_2 of 0 in E, there exist $x_1, x_2, \cdots, x_n \in E$ such that

$$T(\xi_0)B \subset \bigcup_{k=1}^{n} \{T(\xi_0)x_k + V_2\},$$

i.e., for any $x \in B$, there exists an x_k such that

$$T(\xi_0)x - T(\xi_0)x_k \in V_2. \tag{4}$$

By Proposition 1, for any $x \in E$, $\xi \to T(\xi)x$ is a continuous function from $(0, \infty)$ into E. In particular, $\xi \to T(\xi)x_k$ are continuous functions for $k = 1, 2, \cdots, n$. We can, therefore, find $\delta > 0$ such that, for $k = 1, 2, \cdots, n$,

$$T(\xi + \eta)x_k - T(\xi)x_k \in V_2 \quad \text{if } |\eta| < \delta,\ a \leqq \xi,\ \xi + \eta \leqq b.$$

Let $x \in B$ and x_k be as in (4). Then

$$\begin{aligned} T(\xi+\eta)x - T(\xi)x &= T(\xi+\eta)x - T(\xi+\eta)x_k + T(\xi+\eta)x_k - T(\xi)x_k \\ &\quad + T(\xi)x_k - T(\xi)x \\ &= T(\xi+\eta-\xi_0)[T(\xi_0)x - T(\xi_0)x_k] \\ &\quad + [T(\xi+\eta)x_k - T(\xi)x_k] \\ &\quad + T(\xi-\xi_0)[T(\xi_0)x_k - T(\xi_0)x] \\ &\in T(\xi+\eta-\xi_0)V_2 + V_2 + T(\xi-\xi_0)V_2 \\ &\subset V_1 + V_1 + V_1 \subset V; \end{aligned}$$

i.e., $[T(\xi+\eta)-T(\xi)]B\subset V$ for $|\eta|<\delta$, $a\leqq\xi+\eta\leqq b$, $\xi_0<a$. This proves the required continuity of the function $\xi\to T(\xi)$.

That $T(\xi)$ is compact for $\xi>\xi_0$ follows from the fact $T(\xi_0)$ is a compact operator and $T(\xi)=T(\xi-\xi_0)T(\xi_0)$ where $T(\xi-\xi_0)$ is a continuous linear map of E into E.

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ON RECURSIVELY DEFINED ORTHOGONAL POLYNOMIALS[1]

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1. **Introduction.** Consider a set $\{P_n(x)\}$ of orthogonal polynomials defined by the classical recurrence formula,

$$\begin{aligned} &P_n(x) = (x-c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x) \qquad (n=1,2,3,\cdots), \\ &P_{-1}(x)=0, \qquad P_0(x)=1, \qquad c_n \text{ real}, \qquad \lambda_{n+1}>0. \end{aligned} \tag{1.1}$$

In [2], the author initiated a study of (1.1) based on the chain sequences of Wall [6], the fundamental relation being that the zeros

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