DIFFERENTIAL OPERATORS APPROXIMATELY OF THE FORM $(1-(1/W(x))d/dx)^n$

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1. **Introduction.** This paper studies the factorization of certain homogeneous linear differential operators $L=\sum_{i=0}^{n} a_i(x)D^i$ into products $b_0(x)(1-b_1(x)D)\cdots(1-b_n(x)D)$ of first-order factors, the functions a_i and b_i discussed in terms of asymptotic behavior as x tends to infinity in the complex plane. Among other results, we obtain information about the asymptotic behavior of the solutions of $L(y)=0$. Speaking loosely, the operators L are assumed to be expressible in the form $L=(1-W^{-1}D)^n+E_n(1-W^{-1}D)^n+E_{n-1}(1-W^{-1}D)^{n-1}+\cdots+E_1(1-W^{-1}D)+E_0$ where each $E_i\to 0$ as $x\to\infty$, while $\int_{x_0}^{x} W\to\infty$ as $x\to\infty$. In the factorization $L=b_0(1-b_1D)\cdots(1-b_nD)$ we obtain, it will be the case that $b_0\to 1$ and $b_i/(W^{-1})\to 1$ $(i>0)$ as $x\to\infty$.

We use Strodt's asymptotic relations $\prec$, $\sim$, $\approx$, referring to a sort of sectorial neighborhood system of infinity in the complex plane. This system is denoted $\overline{F}(\alpha,\beta)$. (For an index of notation and terminology, see [2, Part IX, pp. 105–107].) Roughly, $f\prec 1$ means $f\to 0$; $f\prec g$ means $f/g\to 0$; $f\sim g$ means $f/g\to 1$, or $(f/g)-1\prec 1$; and $f\approx g$ means $f/g\to c$, where c is a nonzero complex constant. *Logarithmic monomials* are functions of the form $cx^{m_0}(\log x)^{m_1}(\log_2 x)^{m_2}\cdots(\log_q x)^{m_q}$, with c complex and m_i real, $\log_i$ being the i-fold iteration of log. Such functions are $\prec 1$ and $\succ 1$ according as the first nonzero m_i is negative or positive. A *logarithmic domain* is a set of functions which, together with all finite linear combinations thereof (with logarithmic monomials for coefficients), are either $\sim$ logarithmic monomials or else are *trivial* ($\prec$ every logarithmic monomial). The restriction, in Theorem I, to a logarithmic domain $\mathfrak{D}$ has the effect of guaranteeing that the differences $V-W$, where $V\sim W$, will be $\precsim$-comparable with $W_0/\int W_0$ (which is $\approx x^{-1}(\log x)^{-1}\cdots(\log_q x)^{-1}$ for some q).

The letters W, W_i, V, U, etc. are used for functions which are $\sim$ logarithmic monomials of the form

$$cx^{-1}(\log x)^{-1}\cdots(\log_{p-1}x)^{-1}(\log_p x)^{-1+\tau}\cdot(\log_{p+1}x)^{a_1}\cdots(\log_{p+s}x)^{a_s},\qquad \tau>0,$$

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so that their $\int_{x_0}^{\infty}$ diverges. (In fact, $\int_{x_0}^{x} f \sim \tau^{-1} x (\log x) \cdots (\log_p x) f(x)$ for any such f. See [3, Part V, Lemma ζ].) logarithmic monomials of this sort form the *divergence class*. For every M in this class and every $E \prec 1$, $(M^{-1}D)E \prec 1$; in particular, $DE \prec x^{-1}(\log x)^{-1} \cdots (\log_q x)^{-1}$ for $q = 0, 1, 2, \cdots$. This fact is used in Lemma 9.

An operator L is *unimajoral* if $L(y, \prec 1$ whenever $y \prec 1$, while $L(y) \sim 1$ whenever $y \sim 1$; the L's we study are unimajoral. It is known [2, §27] that whenever the coefficients $a_i(x)$ of a unimajoral $L = \sum_{i=0}^{n} a_i(x) D^i$ belong to a logarithmic domain and $a_n(x)$ is nontrivial, L can be approximately factored: an *approximate factorization sequence* $W_1 \lesssim W_2 \lesssim \cdots \lesssim W_n$ of logarithmic monomials in the divergence class can be found such that $L = \dot{W}_n \cdots \dot{W}_1 + \sum_{i=1}^{n} E_i \dot{W}_i \cdots \dot{W}_1 + E_0$, where each $E_i \prec 1$ and by $\dot{W}_i$ we mean the (unimajoral) operator $(1 - W_i^{-1} D)$. A similar expression, with different E_i's (but always $\prec 1$), can be written using any sequence of functions $(V_1, V_2, \cdots, V_n)$ with $V_i \sim W_i$ in place of $(W_1, W_2, \cdots, W_n)$. We are interested in sequences $(V_1, V_2, \cdots, V_n)$ for which the corresponding E_i are all 0 for $i < n$; such sequences yield exact factorizations: $L = (1 + E_n) \dot{V}_n \dot{V}_{n-1} \cdots \dot{V}_1$. In [3, Part II], exact factorization is achieved for the case in which $(W_1, W_2, \cdots, W_n)$ is *separated*: $\exp(\int (W_i - W_j)) \prec 1$ or $\succ 1$ for all pairs $i \neq j$. (For a precise definition, see [3, Part II].) This condition permits the determination of an exact factorization sequence $(V_1, V_2, \cdots, V_n)$ from certain quasilinear algebraic differential equations treated in [2].

The present study treats unimajoral operators which possess approximate factorization sequences $W_1 = W_2 = \cdots = W_n$: $L = (\dot{W})^n + \sum_{i=0}^{n} E_i (\dot{W})^i$. The results of [3, Part II], which at first appear useless in case of such extreme lack of separation, are applied to an operator with a separated factorization sequence obtained from L by some changes of variable. Theorem I, below, takes care of operators $L = (\dot{W})^n + \sum_{i=0}^{n} E_i (\dot{W})^i$ whose coefficients (which are in any case $\prec 1$) satisfy $E_i \prec (\int W)^{i-n}$. We call such operators *regular*, this designation being suggested by analogy to the criterion for regular singular points in the classical treatment of linear differential equations.

Throughout this paper, integral signs should be construed as having a fixed lower limit, the upper limit serving as the independent variable. In situations involving $1/\int W$, the domain of that variable will be subject to obvious and harmless restrictions.

2. **Heuristics.** Introduction of the new variable $s = -\int W$ leads to $\dot{W} = 1 + d/ds$, so the operator $L = (\dot{W})^n + \sum_{i=0}^{n} E_i (\dot{W})^i$ becomes $(1 + d/ds)^n + \sum_{i=0}^{n} E_i (1 + d/ds)^i$. It is natural then to set $y = e^{-s} z$ in

the equation $L(y)=0$; this leads to $((d/ds)^n+\sum_{i=0}^n E_i(d/ds)^i)z=0$. In the trivial case where each $E_i\equiv 0$, the substitutions $s=e^t$ and $z(s)=e^{-t}v(t)$ convert the equation $(d/ds)^n z=0$ into

$$e^{-(n+1)t}(-1)^n n!(1-n^{-1}d/dt)(1-(n-1)^{-1}d/dt)\cdots(1-1^{-1}d/dt)v=0,$$

in which we have an exactly factored operator with the separated factorization sequence $(1, 2, \cdots, n)$. The solutions $v(t)$ are, of course, $\approx\exp\int 1$, $\exp\int 2, \cdots, \exp\int n$. In case the E_i are not all zero, the same substitution in the equation $((d/ds)^n+\sum_{i=0}^n E_i(d/ds)^i)z=0$ leads (by straightforward calculation) to a unimajoral equation for which $(1, 2, \cdots, n)$ is an approximate factorization sequence, provided the E_i satisfy the condition $E_ie^{(n-i)t}\prec 1$ $(i=0, 1, \cdots, n)$. Since $(1, 2, \cdots, n)$ is separated, we may apply Theorem II of [**3**] to obtain an exact factorization $\dot{V}_n\dot{V}_{n-1}\cdots\dot{V}_1v(t)=0$ where $V_1\sim 1$, $V_2\sim 2, \cdots, V_n\sim n$, and solutions $v(t)$ which are $\approx\exp\int V_i$, $i=1, 2, \cdots, n$.

3. **Definition.** If $L=(\dot{W})^n+\sum_{i=0}^n E_i(\dot{W})^i$ with $E_i\prec(\int W)^{i-n}$ for $i=0, 1, \cdots, n$, we shall say L is *regular with respect to* W.

4. **Theorem I.** *Let $\mathfrak{D}$ be a logarithmic domain. Let L be a unimajoral operator expressible in the form $(\dot{W}_0)^n+\sum_{i=0}^n E_i(\dot{W}_0)^i$ with each $E_i\prec 1$ and W_0 in the divergence class. Then either there exits no W in $\mathfrak{D}$ such that L is regular with respect to W, or else $\{W\colon W\in\mathfrak{D}$ and L is regular with respect to $W\}$ is precisely one of the equivalence classes in $\mathfrak{D}\cap\{W\colon W\sim W_0\}$ defined by the equivalence relation $\cong$ where $U\cong V$ means $U-V\prec W_0/\int W_0$. If L is regular with respect to W, then L can be expressed as a product of first-order factors: $L=u\dot{V}_n\cdots\dot{V}_1$, where $V_i\cong W$ for each i. Conversely, if L is expressible as such as a product, then L is regular with respect to W. If L is regular with respect to W, the solutions of $L(y)=0$ are generated by a basis consisting of functions $y_0, y_1, \cdots, y_{n-1}$ where $y_0\prec y_1\prec\cdots\prec y_{n-1}$. For each i, $y_i\sim\exp\int W_i$ for a certain $W_i\sim W$, and* $\log(y_i/y_{i+1})\sim-\log\int W$.

Proof. The theorem summarizes the following lemmas.

5. **Lemma.** *Let L be regular with respect to W. Then there exist functions $W_0, W_1, \cdots, W_{n-1}$ satisfying $W_i-W\sim iW/\int W$ for $i>0$, $W_0-W\prec W/\int W$, such that the equation $L(y)=0$ has solutions $y_0, y_1, \cdots, y_{n-1}$ satisfying $y_i\sim\exp\int W_i$, $i=0, 1, \cdots, n-1$.*

Proof. Let $f=s^ne^{-s}$, where $s(x)=-\int^x W$. We have

$$\dot{W}(f)=ns^{n-1}e^{-s},\ (\dot{W})^2(f)=n(n-1)s^{n-2}e^{-s},\ \cdots,\ (\dot{W})^n(f)=n!e^{-s}.$$

Using the relation $\dot{W}(Zz) = (\dot{W}(Z))(\dot{M}(z))$ where $M = WZ^{-1}\dot{W}(Z)$, we obtain

$$W(fz) = ns^{n-1}e^{-s}Q_1(z),\ (W)^2(fz)$$
$$= n(n-1)s^{n-2}e^{-s}Q_2Q_1(z), \cdots, (W)^n(fz) = n!e^{-s}Q_n \cdots Q_1(z)$$

where $Q_i = -(n-i+1)W/\int W$ for each i. Hence $L(fz)$ may be written $n!e^{-s}L'(z)$ where $L' = \dot{Q}_n \cdots \dot{Q}_1 + \sum_{i=0}^{n} (E_i s^{n-i}/(n-i)!)\dot{Q}_i \cdots \dot{Q}_1$. (All this is verified in an elementary way.) Since L is regular with respect to W, all but the leading coefficient in L' are $\prec 1$. L' is thus a unimajoral operator; furthermore, its factorization sequence $(Q_1, \cdots, Q_n)$ is separated. It follows from [**3**, Theorem I] that there exists a representation of L' in the form $v\dot{U}_n \cdots \dot{U}_1$ with $v \sim 1$ and $U_i \sim Q_i$ for each i. According to [**3**, Theorem II], the equation $L'(z) = 0$ has solutions z_j with $z_j \sim \exp \int U_j$, $j = 1, \cdots, n$. The corresponding solutions of $L(y) = 0$ are of the form $e^{-s}s^n z_j$, which may be expressed as the product of a function ≈ 1 and $\exp \int (W + U_j + nW/\int W)$. The parenthesized term serves as the W_i called for in this lemma, taking $i = j-1$, $y_i = c_i e^{-s}s^n z_{i+1}$, c_i a normalizing constant.

6. **Lemma.** *In Lemma* 5 *we have* $y_0 \prec y_1 \prec \cdots \prec y_{n-1}$.

Proof. It is clear that $y_i/y_{i+1} \sim \exp \int (W_i - W_{i+1})$ and that $W_i - W_{i+1} \sim -W/\int W$. Using [**3**, Part V, Lemma ζ], we see that the latter is $\sim -\tau x^{-1}(\log x)^{-1} \cdots (\log_p x)^{-1}$, for a positive real τ and an integer $p \geqq 0$. Our conclusion then follows from [**3**, Part V, Lemma γ].

7. **Lemma.** *In Lemma* 5 *we have* $\log(y_i/y_{i+1}) \sim -\log \int W$.

Proof. The relation $y_i/y_{i+1} \sim \exp \int (W_i - W_{i+1})$ immediately implies

$$y_i/y_{i+1} = (1 + E) \exp \int (W_i - W_{i+1})$$

where $E \prec 1$. Thus

$$\log(y_i/y_{i+1}) = \int (W_i - W_{i+1}) + \log(1 + E).$$

Since $W_i - W_{i+1} \sim -W/\int W$, $\int (W_i - W_{i+1}) \sim -\log \int W$ (this is seen with the help of [**3**, Part V, Lemma ζ]), which is $\succ 1$. The term $\log(1+E)$ is seen to be $\prec 1$, using [**1**, §27]. Hence we have $\log(y_i/y_{i+1}) \sim -\log \int W$.

8. **Lemma.** *If L is regular with respect to W, there exist functions $V_1, V_2, \cdots, V_n$ such that $V_1 \sim V_2 \sim \cdots \sim V_n \sim W$, $V_i - W \prec W/\int W$ for each i, and $L = u\dot{V}_n \cdots \dot{V}_1$ with $u \sim 1$.*

PROOF. As in the proof of Lemma 5, we have $L(y)=n!e^{-s}v\dot{U}_n \cdots \dot{U}_1(y/f)$. Writing $U_1=-n(1+h_1)W/\int W$ with $h_1\prec 1$, we have $\dot{U}_1(y/f)$

$$=(\dot{U}_1(1/f)\dot{V}_1(y) \text{ where } V_1=U_1(\dot{U}_1(1/f))f \text{ and}$$

$$\begin{aligned}\dot{U}_1(1/f) &= \dot{U}_1(s^{-n}e^s)\\ &= s^{-n}e^s + s(n(1+h_1)Ds)^{-1}(-ns^{-n-1}e^sDs + s^{-n}e^sDs)\\ &= f^{-1}(1+h_1)^{-1}(h_1+s/n).\end{aligned}$$

Thus $V_1=(-nDs/s)(1+h_1)f^{-1}(1+h_1)^{-1}(h_1+s/n)f=-Ds(h_1n/s+1)$ $=W(1-h_1n/\int W)$. From this we see that $V_1-W\prec W/\int W$.

Now $\dot{U}_1(1/f)=ws^{-(n-1)}e^s$ where $w\sim 1/n$. Operating on $(\dot{U}_1(1/f))\dot{V}_1$ with $\dot{U}_2$, we get $(\dot{U}_2(w))\dot{U}_2'(s^{-(n-1)}e^s\dot{V}_1)$ where $U_2'\sim U_2\sim -(n-1)Ds/s$. Writing $\dot{U}_2'=-(n-1)(Ds/s)(1+h_2)$ with $h_2\prec 1$, and proceeding exactly as above, we obtain $\dot{U}_2\dot{U}_1(y/f)=w's^{-(n-2)}e^s\dot{V}_2\dot{V}_1(y)$ where $w'\sim(n(n-1))^{-1}$ and $V_2=W(1-h_2(n-1)/\int W)$. Continuing in this way, we eventually transform $\dot{U}_n \cdots \dot{U}_1(y/f)$ into $w''e^s\dot{V}_n \cdots \dot{V}_1(y)$, where $w''\sim 1/n!$ and $V_i-W\prec W/\int W$ for each i. This gives us $L=u\dot{V}_n \cdots \dot{V}_1$, concluding the proof.

9. **Lemma.** *If $V\sim W$ and $V-W\prec W/\int W$ (equivalently: $V-W \prec V/\int V$), then L is regular with respect to W iff L is regular with respect to V.*

PROOF. Assume $V-W\prec W/\int W$. Then

$$\dot{W} = (V/W)\dot{V} + (1 - V/W) = (1+g)\dot{V} - g$$

where $g=(V-W)/W\prec 1/\int V$. Thus $\dot{W}$ is regular with respect to V. One proves by induction that the operators $(\dot{W})^p$ $(p=2, 3, \cdots)$ are regular with respect to V: If true for $p=k-1$, we have $(\dot{W})^{k-1} =(\dot{V})^{k-1}+\sum_{i=0}^{k-1} F_i(\dot{V})^i$ with $F_i\prec(\int V)^{i-(k-1)}$ and

$$(\dot{W})^k=((1+g)\,\dot{V}-g)\left((\dot{V})^{k-1}+\sum_{i=0}^{k-1}F_i(\dot{V})^i\right).$$

That the latter is regular with respect to V follows from the fact that $\dot{V}(F_i(\dot{V})^i(y))=F_i(\dot{V})^{i+1}(y)-(DF_i/V)(\dot{V})^i(y)$ where $F_i\prec(\int V)^{i+1-k}$ by the induction hypothesis, and $DF_i/V\prec(\int V)^{i-k}$. The latter inequality is seen by writing $F_i=h(\int V)^{i+1-k}$ with $h\prec 1$ and noting that $Dh/V \prec(\int V)^{-1}$ (because $Dh\prec x^{-1}(\log x)^{-1} \cdots (\log_q x)^{-1}$ if $h\prec 1$). The regularity of L with respect to V follows readily from the regularity of the $(\dot{W})^p$ with respect to V and the assumption that L is regular with respect to W. The converse follows from the symmetry of V and W in this discussion.

10. **Lemma.** *If* $L=u\dot{V}_n \cdots \dot{V}_1$ *where* $u \sim 1$, $V_1 \sim V_2 \sim \cdots \sim V_n \sim W$ *and* $V_i - W \prec W/\int W$ *for each* i, *then* L *is regular with respect to* W.

PROOF. With the operators $\dot{V}_i$ expressed in the form $(1+g_i)\dot{W}-g_i$ where $g_i = 1 - W/V_i \prec 1/\int W$, the proof of this lemma closely parallels the proof of Lemma 9.

11. **Lemma.** *If* L *is regular with respect to* W, *and* $V \sim W$ *but* $V-W \gtrapprox W/\int W$, *then* L *is not regular with respect to* V.

PROOF. Suppose L is regular with respect to V. As a consequence of Lemmas 5 and 6, there exist solutions $y_0 \prec y_1 \prec \cdots \prec y_{n-1}$ with $y_i \sim \exp \int W_i$ and $W_i - W \sim iW/\int W$ for $i>0$, while $W_0 - W \prec W/\int W$. At the same time we have solutions $\bar{y}_0 \prec \bar{y}_1 \prec \cdots \prec \bar{y}_{n-1}$ with $\bar{y}_i \sim \exp \int V_i$, $V_i - V \sim iV/\int V$ for $i>0$, while $V_0 - V \prec V/\int V$. Since the y_i and the $\bar{y}_i$ constitute two bases for the space of solutions of $L(y)=0$, it follows readily that $y_0 \approx \bar{y}_0$, whence $y_0/\bar{y}_0 \approx 1$ and $\log(y_0/\bar{y}_0) \lessapprox 1$. But $y_0/\bar{y}_0 \sim \exp \int (W_0 - V_0)$, and $W_0 - V_0 = (W_0 - W) + (W - V) + (V - V_0) \gtrapprox W/\int W$. From this it follows that $\log(y_0/\bar{y}_0) \sim \int (W_0 - V_0) \succ 1$, a contradiction. Therefore L is not regular with respect to V.

12. **Remark.** There exist unimajoral operators L of the form $(\dot{W})^n + \sum_{i=0}^{n} E_i(\dot{W})^i$ such that L is not regular with respect to V for any V.

For example, $(1-(1+1/x)^{-1}D)(1-D)$ can be expressed in the form $(\dot{W})^2 + E_1\dot{W} + E_0$, with E_0, $E_1 \prec 1$, for $W=1$ (indeed, for any $W \sim 1$). That this L is not regular with respect to V for any $V \sim 1$ may be seen as follows: Suppose L is regular with respect to $V \sim 1$. Then by Lemma 5 there exist W_0 and W_1 with $W_0 - V \prec V/\int V$ and $W_1 - V \sim V/\int V$ such that the solutions of $L(y)=0$ are generated by $y_0 \sim \exp \int W_0$ and $y_1 \sim \exp \int W_1$. In this case $W_1 - W_0 \sim V/\int V$ and $V/\int V \sim 1/x$, so $y_1/y_0 \sim e^f$ where $f \sim \log x$, and $\log(y_1/y_0) \sim \log x$. But one sees directly that the solutions of $L(y)=0$ are generated by e^x and x^2e^x, so y_1/y_0 must be $\approx x^2$, whence $\log(y_1/y_0) \sim 2 \log x$, a contradiction.

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