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ON THE EIGENVALUES OF A MATRIX WHICH COMMUTES WITH ITS DERIVATIVE

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Let $V(t)$ be an $n\times n$ matrix whose elements are differentiable functions of t. Epstein [1] has obtained (Theorem 1) necessary and sufficient conditions for $V(t)$ to commute with its derivative $\dot{V}(t)$ in some interval provided that the Jordan canonical form of $V(t)$ maintains the same form throughout the interval (see the definition below). Using this result we show in Theorem 2, under the same restriction, that if $V(t)$ commutes with $\dot{V}(t)$, then the eigenvalues of $\dot{V}(t)$ are the derivatives of the eigenvalues of $V(t)$.

DEFINITION. *Let $S(J)$ be the set of all $n\times n$ matrices $V(t)$ defined in the interval $I: t_1\leqq t\leqq t_2$ and having the properties:*

(i) *the elements $V_{ij}(t)$ of $V(t)$ are differentiable functions in I,*

(ii) *for each $V(t)\in S(J)$ there exists a nonsingular differentiable matrix $P(t)$ such that $V=P^{-1}JP$ for $t\in I$ where J is the Jordan canonical matrix*

$$J = \begin{pmatrix} C_1(t) & 0 & \cdot\ \cdot\ \cdot & 0 \\ 0 & C_2(t) & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & 0 \\ 0 & \cdot & 0 & C_r(t) \end{pmatrix} \tag{1}$$

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and

$$C_k(t) = \lambda_k(t)I_k + E_k, \qquad k = 1, 2, \cdots, r \tag{2}$$

where $C_k(t)$ *is an* $n_k \times n_k$ *matrix,* n_k *being the multiplicity of the eigenvalue* $\lambda_k(t)$ *of* V; I_k *is the* $n_k \times n_k$ *identity matrix and* E_k *is an* $n_k \times n_k$ *matrix all of whose elements equal* 0 *except possibly on the superdiagonal where some elements may equal* 1,

(iii) E_k *and* n_k *are constant for* $t \in I$.

(iv) $\lambda_k(t)$ *are differentiable in* I,

(v) *if* $j \neq k$ *then* $\lambda_j(t) - \lambda_k(t) \neq 0$ *for* $t \in I$.

In the following $[X, Y]$ stands for the Lie bracket: $[X, Y] = XY - YX$.

THEOREM 1 (EPSTEIN [1]). *If* $V(t) \in S(J)$ *and* $V = P^{-1}JP$ *then* $[V, \dot{V}] = 0$ *in* I *if and only if*

$$[J, [J, \dot{P}P^{-1}]] = 0, \qquad t \in I.$$

PROOF. Setting $X = \dot{P}P^{-1}$ we have by direct calculation

$$\begin{aligned} \dot{V} &= P^{-1}(\dot{J} + [J, X])P, \\ [V, \dot{V}] &= P^{-1}([J, \dot{J}] + [J, [J, X]])P. \end{aligned} \tag{3}$$

Since $[J, \dot{J}] = 0$, the theorem follows.

We need the following lemma (see Jacobson [2]).

LEMMA. *If* A *is an* $n \times n$ *matrix and* X *is any solution of*

$$[A, [A, X]] = 0$$

then $[A, X]$ *is nilpotent.*

THEOREM 2. *If* $V(t) \in S(J)$ *and* $[V, \dot{V}] = 0$ *in* I, *then the eigenvalues of* $\dot{V}(t)$ *are the derivatives of the eigenvalues of* $V(t)$ *for* $t \in I$.

PROOF. If $V = P^{-1}JP$ then from Theorem 1, J must satisfy

$$[J, [J, X]] = 0$$

where $X = \dot{P}P^{-1}$. We partition X in accordance with (1), that is $X = (X_{ij})$ where X_{ij} is of order $n_i \times n_j$. Using (2) we see that X_{ij} must satisfy

$$C_i^2 X_{ij} - 2C_i X_{ij} C_j + X_{ij} C_j^2 = 0, \qquad i, j = 1, 2, \cdots, r. \tag{4}$$

The left-hand side of (8) defines a linear operator T acting on X_{ij}. The eigenvalues of T are $(\lambda_i - \lambda_j)^2$ (see Hausner [3]). Therefore if $i \neq j$ the only solution of (4) is $X_{ij} = 0$. Consequently X consists only of diagonal blocks X_{ii} which satisfy

$$[E_i, [E_i, X_{ii}]] = 0, \qquad i = 1, 2, \cdots, r.$$

It follows from the Lemma that $[E_i, X_{ii}]$ is nilpotent for $i = 1, \cdots, r$. We note that the matrix $[J, X]$ consists only of the nilpotent diagonal blocks $[E_i, X_{ii}]$. From (3) we see that $\dot{V}$ has the same eigenvalues as $\dot{J} + [J, X]$, namely the roots of

$$\det([J, X] + \dot{J} - \lambda I) = 0.$$

Since $[J, X]$ consists only of the diagonal blocks $[E_i, X_{ii}]$ and $\dot{J}$ consists only of the diagonal blocks $\dot{\lambda}_i(t) I_i$, we have

$$\begin{aligned} \det([J, X] + \dot{J} - \lambda I) &= \prod_{i=1}^{r} \det([E_i, X_{ii}] + (\dot{\lambda}_i - \lambda) I_i) \\ &= \prod_{i=1}^{r} (\dot{\lambda}_i - \lambda)^{n_i} \end{aligned}$$

since $[E_i, X_{ii}]$ is nilpotent. Therefore the eigenvalues of $\dot{V}$ are just $\dot{\lambda}_i$ and the theorem is proved.

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