METRIC-DEPENDENT DIMENSION FUNCTIONS[1]

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The starting point of this study was an attempt to solve a problem proposed by G. M. Rosenstein [3] in his doctoral dissertation.

ROSENSTEIN'S PROBLEM. Suppose that R is a metrizable space, $\dim R \geqq n$ (covering dimension), and ρ is any compatible metric for R. Then is it true that there exist n pairs of closed sets $C_1, C_1', \cdots, C_n, C_n'$ such that (i) $\rho(C_i, C_i') > 0$ for all i and (ii) if for each i, B_i is a closed set separating C_i and C_i', then $\bigcap_{i=1}^n B_i \neq \varnothing$?

REMARK. If, in this problem, "(i) $\rho(C_i, C_i') > 0$" is replaced by "(i*) $C_i \cap C_i' = \varnothing$," one obtains a characteristic property for metric spaces R such that $\dim R \geqq n$. (See [1, Remark, for the separable case, p. 78].)

Our main result is an example which shows that the answer to the problem is in the negative. In the notation given below, our example is a metric space (R, ρ) such that $\dim R = 2$ and $d_2(R, \rho) = 1$. For $n = 1$ the problem is answered in the affirmative.

DEFINITION OF $d_1(R, \rho)$. Let (R, ρ) be a metric space with metric ρ. We define $d_1(R, \rho)$ inductively as follows: For the empty set $\varnothing$, $d_1(\varnothing, \rho) = -1$. If for every pair of closed subsets F, H of R with $\rho(F, H) > 0$ there exists an open set G with $F \subset G \subset R - H$ and with $d_1(\overline{G} - G, \rho^*) \leqq n - 1$, where ρ^* is the restriction of ρ to $\overline{G} - G$, then we say $d_1(R, \rho) \leqq n$. If there is no such integer n, then we say $d_1(R, \rho) = \infty$.

THEOREM. *For any metric space* (R, ρ) *we have* $d_1(R, \rho) = \operatorname{Ind} R$, *where* Ind R *is the large inductive dimension of* R *defined by means of sets separating disjoint closed pairs of subsets.*

PROOF. It is evident that $d_1(R, \rho) \leqq \operatorname{Ind} R$. When $d_1(R, \rho) = \infty$, it is also evident that $d_1(R, \rho) \geqq \operatorname{Ind} R$. Hence we suppose that $d_1(R, \rho) \leqq n$ and make the induction assumption that $d_1(R', \rho') \geqq \operatorname{Ind} R'$ for any metric space (R', ρ') with $d_1(R', \rho') \leqq n - 1$. Let H and F be disjoint closed subsets of R. Put $D_i = \{x : \rho(x, H) < 1/i\}$, and $E_i = \{x : \rho(x, F) < 1/i\}$, $i = 1, 2, \cdots$. Then there exist open sets M_i with $H \subset M_i \subset D_i$, $d_1(\overline{M}_i - M_i, \rho) \leqq n - 1$, $i = 1, 2, \cdots$, and open sets N_i with $F \subset N_i \subset E_i$, $d_1(\overline{N}_i - N_i, \rho) \leqq n - 1$, $i = 1, 2, \cdots$. Put $G_i = M_i - \overline{N}_i$, $i = 1, 2, \cdots$, and $G = \bigcup_{i=1}^\infty G_i$. Then it can easily be seen that $H \subset G \subset \overline{G} \subset R - F$.

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Since $\overline{G}_i - G_i \subset (\overline{M}_i - M_i) \cup (\overline{N}_i - N_i)$, we have $\operatorname{Ind}(\overline{G}_i - G_i) \leqq n-1$ by the induction assumption. Since $\{G_i : i=1, 2, \cdots\}$ is locally finite at every point of $R-(H\cup F)$ and $\overline{G}-G$ is contained in $R-(H\cup F)$, we have $\overline{G}-G \subset \bigcup \{\overline{G}_i - G_i : i=1, 2, \cdots\}$. Every point x of $\overline{G}-G$ has a relative neighborhood $U(x)$ which is contained in the sum of a finite number of elements of $\{\overline{G}_i - G_i : i=1, 2, \cdots\}$. Hence $\operatorname{Ind} U(x) \leqq n-1$ and we have $\operatorname{Ind}(\overline{G}-G) \leqq n-1$ by the local dimension theorem. Thus we know $\operatorname{Ind} R \leqq n$ and the theorem is proved.

COROLLARY. *If R is a metrizable space with* $\dim R = 1$ *and ρ is any compatible metric for R, then there exists a pair of closed sets C and C' with $\rho(C, C') > 0$ such that the empty set cannot separate C and C'.*

DEFINITION OF $d_2(R, \rho)$. Let (R, ρ) be a metric space with metric ρ. We write $d_2(\varnothing, \rho) = -1$. If there exists a greatest integer n such that there exist n pairs $C_1, C_1', \cdots, C_n, C_n'$ such that (i) $\rho(C_i, C_i') > 0$ and (ii) if for each i, B_i is a closed set separating R between C_i and C_i', then $\bigcap_{i=1}^n B_i \neq \varnothing$, then we say $d_2(R, \rho) = n$. Otherwise $d_2(R, \rho) = \infty$.

Rosenstein's problem may now be stated as follows: Is it true that $d_2(R, \rho) = \dim R$?

EXAMPLE. In the closed 3-cell I^3 we define a countable disjoint collection $\mathfrak{A} = \{A_i : i=1, 2, \cdots\}$, set $A = \bigcup \mathfrak{A}$ and $R = I^3 - A$. We prove that $\dim R = 2$ and $d_2(R, \rho) = 1$, where ρ is the Euclidean metric on I^3. Let $\mathfrak{U} = \{U_1, U_2, \cdots\}$ be any countable base for the topology of I^3. The set of all finite unions of elements of $\mathfrak{U}$ is countable; hence so is the set of all quadruples of such unions and, a fortiori, any subset. Thus there exists a countable sequence $Q_1, Q_2, \cdots$, such that for all i

(1) $Q_i = \{B_i, C_i, D_i, E_i\}$,

(2) each of B_i, C_i, D_i, E_i is a finite union of elements of $\mathfrak{U}$,

(3) $\rho(B_i, C_i) > 0$ and $\rho(D_i, E_i) > 0$, and

(4) if $Q' = \{B', C', D', E'\}$ is any quadruple satisfying (2) and (3) then for some i we have $Q' = Q_i$.

For each i we define closed sets M_i and N_i such that M_i separates B_i and C_i, N_i separates D_i and E_i, and we set $A_i = M_i \cap N_i$. We want $\mathfrak{A} = \{A_i : i = 1, 2, \cdots\}$ to be a disjoint collection. Let $\epsilon_i = \min\{\rho(B_i, C_i), \rho(D_i, E_i)\}$. Let $\pi_1, \pi_1', \pi_2, \pi_2', \cdots$ be a monotonically increasing sequence of prime numbers such that for all i, $1/\pi_i < \epsilon_i/\sqrt{3}$. Divide I^3 into π_i^3 small closed cubes whose edges have length $1/\pi_i$, let H_i denote the union of all such cubes that intersect B_i, and let M_i = boundary of H_i. Similarly, using π_i' instead of π_i, let K_i be the union of all small cubes intersecting D_i and let N_i = boundary of K_i. Set $A_i = M_i \cap N_i$. If $x \in A_i$, then one coordinate of x is of the form a/π_i and one coordinate is of the form b/π_i', where a and b are integers, $0 < a < \pi_i$, $0 < b < \pi_i'$.

ASSERTION 1. For all i, M_i is a closed set separating B_i and C_i and N_i is a closed set separating D_i and E_i.

ASSERTION 2. For all i, A_i is closed and $\dim A_i \leqq 1$.

PROOF. If a and b are positive integers with $0 < a < \pi_i$, $0 < b < \pi_i'$, then $a/\pi_i \neq b/\pi_i'$; hence if $x \in A_i$, then at least two of its coordinates are rational. By [1, Example III 6, p. 29] this shows that $\dim A_i \leqq 1$.

ASSERTION 3. If $i \neq j$, then $A_i \cap A_j = \emptyset$.

PROOF. Assume there exists $x \in A_i \cap A_j$. Then there exist four integers $0 < a < \pi_i$, $0 < b < \pi_i'$, $0 < c < \pi_j$, $0 < d < \pi_j'$ such that x must have coordinates equal to each of a/π_i, b/π_i', c/π_j, d/π_j', four distinct numbers. But x has only 3 coordinates.

ASSERTION 4. $d_2(R, \rho) \leqq 1$.

PROOF. Let B, C, D, E be any relatively closed sets in R such that $\rho(B, C) > 0$, $\rho(D, E) > 0$. Then their closures in I^3 are compact and for some i we have $B \subset B_i$, $C \subset C_i$, $D \subset D_i$ and $E \subset E_i$; so M_i separates B and C in I^3, N_i separates D and E in I^3. But then $M_i \cap R$ and $N_i \cap R$ are corresponding separating sets in R, and their intersection is vacuous since $M_i \cap N_i = A_i \subset I^3 - R$.

ASSERTION 5. $\dim R \leqq 2$.

PROOF. This follows from Brouwer's theorem on invariance of domain, since it can easily be seen that A is dense in I^3.

ASSERTION 6. $\dim R \geqq 2$.

To prove this assertion we need the following two lemmas.

LEMMA 1. *Let S be a subset of the closed n-cell I^n with* $\dim S \leqq n-2$. *Then for any points p and q in $I^n - S$ there exists a continuum K such that* (i) *K is contained in $I^n - S$ and* (ii) *K contains p and q.* (*See* [4].)

LEMMA 2. *A continuum cannot be decomposed into a countably infinite or finite* (*but more than one*) *union of pairwise disjoint closed subsets.* (*See* [2, *Theorem* 44, *p.* 30].)

PROOF OF ASSERTION 6. Assume that $\dim R < 2$. If A is closed, then R is an F_σ and $\dim I^3 \leqq \max(\dim A, \dim R) < 2$, which is impossible. Thus there exist integers i and j with $i \neq j$ such that $A_i \neq \emptyset$, $A_j \neq \emptyset$. Take $p \in A_i$, $q \in A_j$. By Lemma 1 there exists a continuum K such that (i) $\{p, q\} \subset K$ and (ii) $K \subset I^3 - R = A$. Thus $K = \bigcup_{i=1}^\infty (K \cap A_i)$, a countable union of pairwise disjoint closed sets at least two of which are not vacuous. But this is impossible by Lemma 2, so the proof of Assertion 6 is completed.

ASSERTION 7. $d_2(R, \rho) \geqq 1$.

PROOF. Since dim $R=2$, $d_2(R, \rho) \neq -1$. It is obvious that $d_2(R, \rho) = 0$ if and only if $d_1(R, \rho) = 0$. On the other hand we already know that $d_1(R, \rho) = \dim R$ by the above theorem. Thus $d_2(R, \rho) < 1$ contradicts dim $R=2$.

REMARK. By a construction quite similar to that just given, one may start with I^n for $n > 3$, and define a subset R such that dim $R = n-1$ and $d_2(R, \rho) \leqq n/2$.

PROBLEM. Is it true that, given a metric space (R, ρ), dim $R \geqq n$ implies that $d_2(R, \rho) \geqq n/2$?

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