

MEROMORPHIC FUNCTIONS ON CERTAIN RIEMANN SURFACES

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In the present paper we shall introduce four new principles for the classification of Riemann surfaces based on bounded analytic functions. Our principal aim is to establish an analogue of Heins' composition theorem [1].

1. Throughout the paper we shall denote by R a Riemann surface. For a domain Ω in R , we represent by $AB(\Omega)$ the class of all the single-valued bounded analytic functions on the closure $\bar{\Omega}$. For a meromorphic function f on a domain Ω , we use the notation $\nu(w; f, \Omega)$ to express the number of times that f attains w in Ω .

DEFINITION 1. We say that $R \in \mathfrak{M}_B$ if the maximum principle $\sup_{p \in \Omega} |f(p)| = \sup_{p \in \partial\Omega} |f(p)|$ holds for every f in the class $AB(\Omega)$ for every $\Omega \subset R$ with compact relative boundary $\partial\Omega$.

DEFINITION 2. We say that $R \in \mathfrak{A}_B$ if, for every $\Omega \subset R$ with compact $\partial\Omega$, every function in the class $AB(\Omega)$ has its limit at every ideal boundary point in the sense of Kerékjártó-Stoilow.

DEFINITION 3. We say that $R \in \mathfrak{D}_B$ if, for every $\Omega \subset R$ with compact $\partial\Omega$, the cluster set $L_f(\Omega)$ of every f in the class $AB(\Omega)$ attached to the ideal boundary is a totally disconnected set in the complex w -plane.

From the definitions, we have immediately, $\mathfrak{D}_B \subset \mathfrak{A}_B \neq O_{AB}$, $\mathfrak{D}_B \neq O_{AB}$ and $\mathfrak{M}_B \subsetneq O_{AB}$. If R has only a countable number of ideal boundary points, we see easily that $R \in \mathfrak{A}_B$ implies $R \in \mathfrak{D}_B$. It is also evident that, if the surfaces are restricted to plane domains, then $\mathfrak{M}_B = O_{AB} \subset \mathfrak{D}_B \subset \mathfrak{A}_B$.

THEOREM 1. $\mathfrak{D}_B \subset \mathfrak{M}_B \subsetneq \neq O_{AB}$ in general. In the planar case, $\mathfrak{D}_B = \mathfrak{M}_B = O_{AB}$.

PROOF. We suppose that there exists an $R \in \mathfrak{D}_B$ such that $R \notin \mathfrak{M}_B$. There exists an $f \in AB(\Omega)$ on an Ω with compact $\partial\Omega$ which satisfies $|f(p_0)| > \sup_{p \in \partial\Omega} |f(p)|$ at a point $p_0 \in \Omega$. Then the boundary $\partial f(\Omega)$ of the domain $f(\Omega)$ must contain a continuum K which is disjoint from $f(\partial\Omega)$. Every $w^* \in K$ satisfies $\nu(w^*; f, \Omega) = 0$. On the other hand, for every $w^* \in K$, there exists a sequence $\{w_k\}$ such that $w_k \rightarrow w^*$ and

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$\nu(w_k; f, \Omega) \geq 1$. We can choose a sequence $\{p_{k_n}\}$, with $f(p_{k_n}) = w_{k_n}$, which tends to an ideal boundary point φ^* . This shows that every $w^* \in K$ is a limiting value of f at an ideal boundary point of R , which contradicts the total-disconnectedness of the cluster set $L_f(\Omega)$. We conclude that $\mathfrak{D}_B \subset \mathfrak{M}_B$. For the planar case, we have mentioned $\mathfrak{D}_B \supset O_{AB} = \mathfrak{M}_B$ already, so that we conclude $\mathfrak{D}_B = \mathfrak{M}_B = O_{AB}$.

2. Consider a domain D on a Riemann surface, assume that ∂D is analytic, and let f be a meromorphic function on D . We denote by $\Phi(f, D)$ the covering surface over the complex w -plane, equivalent conformally to D by the mapping $f: D \rightarrow \Phi(f, D)$.

For $w \in f(D)$ with $w \notin f(\partial D)$, we let $\{p_n\}_{n=1}^\infty = f^{-1}(w)$ and take $C_\epsilon(w) = \{\zeta \mid |\zeta - w| < \epsilon\}$ with $C_\epsilon(w) \cap f(\partial D) = \emptyset$. If p_n is not a multiple point of f , we take a simply connected domain $D_n \subset D$ which contains w_n and is such that $\Phi(f, D_n)$ is schlicht over $C_\epsilon(w)$. To this end, we continue analytically the function element of f^{-1} with $f^{-1}(w) = p_n$ from the point w along all the rays until we meet singularities or the circumference of $C_\epsilon(w)$; here the analytic continuation is in the sense of Weierstrass, so that a branch point is regarded as a singularity. The collection of all the function elements so obtained is the desired $\Phi(f, D_n)$, and its image under f^{-1} is the D_n . The projection of $\Phi(f, D_n)$ onto the w -plane is a star domain in the sense of Gross, which is a disk $C_\epsilon(w)$ with radial incisions. We remark that $r(\theta)$, the distance between w and the boundary point $w + re^{i\theta}$ of this star domain, is lower-semicontinuous.

In the case where p_n is a multiple point of f , a small modification is needed. Let ρ be the order of multiplicity. We cut $C_\epsilon(w)$ along a radius in advance. By the cut the multiple-valued function element of f^{-1} with $f^{-1}(w) = p_n$ is split into ρ ordinary elements. On performing the same operation as above with respect to these ρ elements in the cut disk, we obtain the domains D_{n_j} ($j=1, \dots, \rho$) in exactly the same way. The domain D_n , which is defined as the union of $\bigcup_{j=1}^\rho D_{n_j}$ and the ρ counter-images of the cut, contains p_n as an interior point. We define the lower-semicontinuous $r(\theta)$ similarly.

It is to be noted that all the D_n are mutually disjoint.

DEFINITION 4. We say that $R \in \mathfrak{D}'_B$ if every $f \in AB(\Omega)$ for every Ω with compact analytic $\partial\Omega$ has the following property: For every $w \notin f(\Omega) - f(\partial\Omega)$, every ϵ with $C_\epsilon(w) \cap f(\partial\Omega) = \emptyset$, and every $p_n \in f^{-1}(w)$, $r(\theta) = \epsilon$ holds with the exception of at most a totally disconnected closed set in $0 \leq \theta \leq 2\pi$.

THEOREM 1'. $\mathfrak{D}'_B \subset \mathfrak{M}_B \subset \neq O_{AB}$ save in the planar case, where $\mathfrak{D}'_B \subset \mathfrak{D}_B = \mathfrak{M}_B = O_{AB}$.

PROOF. Suppose there exists an $R \in \mathcal{D}'_B$ such that $R \notin \mathcal{M}_B$. Let $f \in \text{AB}(\Omega)$ be the one violating the maximum principle, as in the proof of Theorem 1. Then there exists an unbounded component of $(f(\partial\Omega))^c$ which contains a continuum $K \subset \partial f(\Omega)$; here the superscript c expresses the complement. Let \emptyset be the maximal domain contained in the set $\{w \mid \nu(w; f, \Omega) = 0\}$. We may require K to be contained in $\partial\emptyset$. Every $w^* \in K$ satisfies $\nu(w^*; f, \Omega) = 0$. On the other hand, there is a point $w_0 \in f(\Omega)$ in a neighborhood of w^* such that $C_\epsilon(w_0) \cap f(\partial\Omega) = \emptyset$ and $w^* \in C_\epsilon(w_0)$ for an $\epsilon > 0$. Then $C_\epsilon(w_0) \cap K$ contains a continuum. Since $K \subset \partial\emptyset$, we conclude that the star domain, being the projection of the $\Phi(f, D_n)$ for an n with center at w_0 , has the following property: The arguments of the incisions form a totally-disconnected set in $0 \leq \theta \leq 2\pi$ and, even disregarding the incisions emanating from algebraic branch points, the union of incisions contains a continuum. This must hold, however, at any w in a neighborhood of w_0 , which is impossible. Thus we get $\mathcal{D}'_B \subset \mathcal{M}_B$. The remaining assertions have been verified already.

3. THEOREM 2. *If $R \in \mathcal{D}_B \cap \mathcal{D}'_B$ and $f \in \text{AB}(\Omega)$ of an $\Omega \subset R$ with compact $\partial\Omega$, then $N_\Omega(f) = \sup_{w \in f(\Omega)} \nu(w; f, \Omega) < \infty$.*

PROOF. First we show that $\nu(w; f, \Omega) < \infty$ at every w . Suppose that there exists a w with $\nu(w; f, \Omega) = \infty$. Let $C_\epsilon(w)$ be disjoint from $f(\partial\Omega)$, let $\{p_n\} = f^{-1}(w)$, and consider the above-mentioned D_n and $\Phi(f, D_n)$. By the assumption, the intersection $I_n(r)$ of the incisions with the circle $C(r) = \{\zeta \mid |\zeta - w| = r\}$, $0 < r < \epsilon$, is closed and totally disconnected in $C(r)$. Since the union of a countable number of totally-disconnected closed sets in a locally compact space is again totally disconnected (see [2]), the union $\bigcup_n I_n(r)$ is totally disconnected in $C(r)$, so that $K(r) = C(r) - \bigcup_n I_n(r)$ contains a continuum. This shows that every function element f^{-1} at w can be continued analytically to every point in $K(r)$, and, therefore, $\nu(w^*; f, \Omega) = \infty$ for every $w^* \in K(r)$. Then w^* is a limiting value of f at an ideal boundary point. Thus we see that a continuum $K(r)$ is contained in $L_f(\Omega)$, contradicting the assumption. We conclude that $\nu(w; f, \Omega) < \infty$ for every $w \notin f(\partial\Omega)$.

If $w \in f(\partial\Omega)$, then we can move $\partial\Omega$ slightly in such a manner that the variation of the valency function ν at w by this process is finite. This is possible, since $\partial\Omega$ is analytic and f is analytic there. We conclude that $\nu(w; f, \Omega) < \infty$ for $w \in f(\partial\Omega)$.

Next, we show the boundedness of $\nu(w; f, \Omega)$. In the w -plane we consider the set $E_n = \{w \mid \nu(w; f, \Omega) \geq n\}$. For every $w^* \in \partial E_n - f(\partial\Omega)$, we have $\nu(w^*; f, \Omega) \leq n-1$. Take a small $C_\epsilon(w^*)$ and consider the previously mentioned $\Phi(f, D_l)$, $l=1, \dots$, $\nu^* = \nu(w^*, f, \Omega) \leq n-1$.

Since $w^* \in \partial E_n$, there are points $w_\mu \in E_n$ such that $w_\mu \rightarrow w^*$ and $\{f^{-1}(w_\mu)\} - D_1 \cup \dots \cup D_r \neq \emptyset$. Choose p_{μ_λ} with $f(p_{\mu_\lambda}) = w_\mu \in \{w_\mu\}$, which tends to an ideal boundary point \wp^* . Then $\lim_{\lambda \rightarrow \infty} f(p_{\mu_\lambda}) = \lim_{\lambda \rightarrow \infty} w_{\mu_\lambda} = w^*$. This shows that $w^* \in L_f(\Omega)$, and, therefore, that $\partial E_n \subset L_f(\Omega)$. By the assumption we conclude that ∂E_n does not contain a continuum.

The total-disconnectedness of ∂E_n for every n shows that $\nu(w; f, \Omega)$ is constant in each component of $(f(\partial\Omega))^c$, which are finite in number. Thus we conclude that $N_\Omega(f) < \infty$.

We remark that it is easy to prove that $L_f(\Omega) \cap (f(\partial\Omega))^c = (\bigcup_{n=1}^\infty E_n) \cap (f(\partial\Omega))^c$.

THEOREM 3. *Let R be in $\mathfrak{D}_B \cap \mathfrak{D}'_B$ and Ω be a subdomain of R with compact analytic $\partial\Omega$. Then, for every $f \in \text{AB}(\Omega)$, the number $N_\Omega(f)$ is finite and the set $L_f(\Omega)$ is totally disconnected and is a union of a countable number of closed $N_{\mathfrak{B}}$ sets. For every meromorphic f with $N_\Omega(f) = \infty$, the set $\text{Cl}[f(\Omega)]$ coincides with the extended plane and the set $\{w \mid \nu(w; f, \Omega) < \infty\}$ is a union of a countable number of $N_{\mathfrak{B}}$ sets.*

PROOF. Let f be meromorphic in $\bar{\Omega}$ and be such that $\text{cl}[f(\Omega)]$ is the extended plane. Let $\{\Omega_k\}_0^\infty$ be such that $\Omega_k \supset \Omega_{k+1}$ and $\bigcap \Omega_k = \emptyset$, where each Ω_k is a union of subends. The set $E_k = [\nu(w; f, \Omega_k) = 0]$ is closed in the extended plane and is such that $E_k \subset E_{k+1}$, $[\nu(w; f, \Omega) < \infty] \subset \lim E_k$. We infer that $E_k \in N_{\mathfrak{B}}$, for otherwise there would exist a nonconstant $\phi \in \text{AB}(E_k^c)$ and $\phi \circ f \in \text{AB}(\Omega_k)$ would violate the maximum principle. Therefore $[\nu(w; f, \Omega) < \infty]$ is a union of a countable number of $N_{\mathfrak{B}}$ sets.

Let f be in the class $\text{AB}(\Omega)$. Let Ω^* be a generic union of a finite number of ends, W_{Ω^*} be the set of all the interior points of $\text{cl}[f(\Omega^*)]$, and $E_0^* = W_{\Omega^*} - f(\Omega^*)$. We have similarly that $E_0^* \in N_{\mathfrak{B}}$. Since $L_f(\Omega) \subset \lim_{\Omega^*} E_0^*$, we conclude that $L_f(\Omega)$ is a union of a countable number of $N_{\mathfrak{B}}$ sets.

4. Let $\{\Omega_i\}$ be a defining sequence of an ideal boundary point \wp . Let the local degree of f at \wp be defined by

$$d(\wp, f) = \min_{\Omega_i} N_{\Omega_i}(f).$$

Then we have the following fact: For an $\text{AB}(\Omega)$ function $f(p)$, there exists a suitable subend Ω_m for which

$$d(\wp, f) = \frac{1}{2\pi} \int_{\partial\Omega_m} \frac{\partial}{\partial n} \log \frac{1}{|f - w_0|} ds, \quad w_0 = f(\wp),$$

and, further, it is possible to choose a suitable subend Ω^* containing \wp on which f is $(1, d(\wp, f))$ with the exception of a totally-disconnected closed set being a countable sum of $N_{\mathfrak{g}}$ sets. The proof of the above facts and the following propositions are carried out along the same line as in Heins' paper [1]. In the following it is always assumed that the surface R belongs to the class $\mathfrak{D}_B \cap \mathfrak{D}'_B$.

PROPOSITION 1. *If f and g are in the class $AB(\Omega)$ and are not both identically zero, then f/g possesses a limit (finite or infinite) at every ideal boundary point.*

PROPOSITION 2. *Let g be an $AB(\Omega_m)$ function for some subend Ω_m satisfying the conditions that:*

- (1) g is $(1, n)$ on Ω_m and $n = \min_{f \in AB(\Omega_m)} d(\wp, f)$,
- (2) the limit of g at \wp is zero,
- (3) $|g| = 1$ on $\partial\Omega_m$.

Let f be an $AB(\Omega_m)$ function satisfying $f(\wp) = 0$ and $|f(p)| < 1$ on Ω_m . Then $|f/g| \leq 1$ on Ω_m . If equality prevails at any point of Ω_m or if the modulus of the limit of f/g at \wp is one, then $f = \epsilon g$, where ϵ is a constant of modulus one.

THEOREM 4. *For any $f \in AB(\Omega_m)$ with $\sup_{\Omega} |f| < 1$, there exists a unique AB function ϕ in the unit disk $|z| < 1$ such that $f = \phi \circ g$ in Ω_m .*

This is an analogue of the so-called Heins composition theorem.

5. Several years ago Kuramochi [3] proved the following: If $R \in O_{HB} - O_G$, then $\Omega \in O_{AB}$. Several authors have generalized this elegant result in various ways. In particular Toda and Matsumoto [6] obtained the following generalization: If $R \in O_{AB}^0 - O_G$ and there exists an ideal boundary point \wp with positive harmonic measure $\omega(p; \wp, \Omega)$, then $\Omega \in O_{AB}$. (For the definition of the class O_{AB}^0 the reader is referred to Kuroda [4], in which it is shown that $O_{HB} \subset O_{AB}^0$. Their proof even yields the following better result:

THEOREM 5. *If $R \in \mathfrak{A}_B - O_G$ and there is an ideal boundary point \wp with positive harmonic measure $\omega(p; \wp, \Omega)$, then $\Omega \in O_{AB}$.*

Concerning our classes, we remark that the following relations have been obtained: $O_{AB}^0 \subset \mathfrak{A}_B$ [6], $O_{AB}^0 \subset \mathfrak{D}_B$ [4], [5].

There remain several unsolved problems. Among them the following two seem to be important. Does \mathfrak{A}_B coincide with \mathfrak{D}_B ? Does \mathfrak{D}_B coincide with \mathfrak{D}'_B ?

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ON THE EIGENVALUES OF A MATRIX WHICH COMMUTES WITH ITS DERIVATIVE

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Let $V(t)$ be an $n \times n$ matrix whose elements are differentiable functions of t . Epstein [1] has obtained (Theorem 1) necessary and sufficient conditions for $V(t)$ to commute with its derivative $\dot{V}(t)$ in some interval provided that the Jordan canonical form of $V(t)$ maintains the same form throughout the interval (see the definition below). Using this result we show in Theorem 2, under the same restriction, that if $V(t)$ commutes with $\dot{V}(t)$, then the eigenvalues of $\dot{V}(t)$ are the derivatives of the eigenvalues of $V(t)$.

DEFINITION. Let $S(J)$ be the set of all $n \times n$ matrices $V(t)$ defined in the interval $I: t_1 \leq t \leq t_2$ and having the properties:

- (i) the elements $V_{ij}(t)$ of $V(t)$ are differentiable functions in I ,
- (ii) for each $V(t) \in S(J)$ there exists a nonsingular differentiable matrix $P(t)$ such that $V = P^{-1}JP$ for $t \in I$ where J is the Jordan canonical matrix

$$(1) \quad J = \begin{pmatrix} C_1(t) & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & C_2(t) & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & C_r(t) & & \end{pmatrix}$$

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