HARNACK'S AND WEYL'S INEQUALITIES[1]

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Let $Z = U_0 R V_0$ in which $R = \operatorname{diag}(r_1, \cdots, r_n)$, $r_i > 0$, $i = 1, \cdots, n$, and U_0 and V_0 are n-square unitary matrices. In [4] S. H. Tung extends Harnack's inequality by a Lagrange multiplier argument to obtain

THEOREM 1. *If U is any n-square unitary matrix and $1 > r_k \geqq 0$, $k = 1, \cdots, n$, then*

$$(1) \quad \prod_{k=1}^{n} \frac{1 - r_k}{1 + r_k} \leqq \frac{\det (I - ZZ^*)}{\det (I - ZU^*) \det (I - UZ^*)} \leqq \prod_{k=1}^{n} \frac{1 + r_k}{1 - r_k}.$$

The numerator of the middle fraction in (1) is just $\prod_{k=1}^{n} 1 - r_k^2$ and hence upon clearing of fractions (1) is directly equivalent to

$$(2) \qquad \prod_{k=1}^{n} 1 + r_k \geqq |\det (I - A)| \geqq \prod_{k=1}^{n} 1 - r_k$$

where $A = RW$ and $W = V_0 U^* U_0$.

What we show here is that some standard techniques [see 3, p. 112] applied to the singular value—eigenvalue inequalities of H. Weyl [5] will produce the following generalization of (2). (Recall that the *singular values* of A are the non-negative square roots of the eigenvalues of A^*A e.g., the singular values of $A = RW$ are the numbers $r_1, \cdots, r_n$.)

THEOREM 2. *Suppose that A has eigenvalues $\lambda_1, \cdots, \lambda_n$, $|\lambda_1| \geqq \cdots \geqq |\lambda_n|$, and singular values $r_1 \geqq \cdots \geqq r_n$. If $r_1 < 1$ and $1 \geqq c_1 \geqq \cdots \geqq c_p > 0$ then if $1 \leqq p \leqq n$,*

$$(3) \quad \begin{aligned} \prod_{k=1}^{p} 1 + c_k r_k \geqq \prod_{k=1}^{p} 1 + c_k |\lambda_k| &\geqq \prod_{k=1}^{p} |1 - c_k \lambda_k| \\ &\geqq \prod_{k=1}^{p} |1 - c_k \lambda_k| \geqq \prod_{k=1}^{p} 1 - c_k r_k. \end{aligned}$$

PROOF. Since

$$(4) \qquad |\lambda_1| \leqq r_1 < 1,$$

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the two middle inequalities in (3) are trivial. Weyl's inequalities are

$$\prod_{k=1}^{p} |\lambda_k| \leqq \prod_{k=1}^{p} r_k \tag{5}$$

with equality for $p=n$. Actually (5) follows right away by applying (4) to the Pth Grassmann compound matrix of A (this was Weyl's proof).

Next observe that the function $\log(1+c_k e^x)$ is convex increasing for all real x and hence the function

$$f(t) = f(t_1, \cdots, t_n) = \sum_{k=1}^{p} \log(1 + c_k e^{t_k}) \tag{6}$$

is convex in Euclidean n-space. By taking logs in (5) and applying a familiar result in Hardy-Littlewood-Pólya [2, p. 49] it follows that

$$\log|\lambda| = S \log r \tag{7}$$

where $\log|\lambda| = (\log|\lambda_1|, \cdots, \log|\lambda_n|)$, $\log r = (\log r_1, \cdots, \log r_n)$ and S is an n-square doubly stochastic (d.s.) matrix. If we set

$$g(S) = f(S \log r) \tag{8}$$

it is then easy to verify that g is convex on the polyhedron of d.s. matrices and thereby assumes its maximum on a vertex of this polyhedron. According to Birkhoff's result [1] these vertices are permutation matrices and by using the increasing properties of f we compute

$$\begin{aligned} \sum_{k=1}^{p} \log(1 + c_k |\lambda_k|) &= \sum_{k=1}^{p} \log(1 + c_k \exp[\log|\lambda_k|]) = f(\log|\lambda|) \\ &= f(S \log r) \leqq f(\log r_1, \cdots, \log r_k) \\ &= \sum_{k=1}^{p} \log(1 + c_k \exp[\log r_k]) \\ &= \sum_{k=1}^{p} \log(1 + c_k r_k). \end{aligned} \tag{9}$$

We can then get the upper inequality in (3) by taking the exponential of the two ends of (9).

The lower inequality in (3) is done in about the same way. This time observe that the function $\log(1-c_k e^x)$ is concave decreasing for all $x<0$. Hence

$$f(t_1, \cdots, t_n) = \sum_{k=1}^{p} \log(1 - c_k e^{t_k}) \tag{10}$$

is concave for $t_k<0$, $k=1, \cdots, n$. Define $g(S)$ by (8), using (10) instead of (6) for the definition of f. Now $g(S)$ is minimal for S a permutation matrix. Thus for some selection $1 \leqq i_1 < \cdots < i_p \leqq n$ we have

$$\begin{aligned}\sum_{k=1}^{p} \log(1 - c_k |\lambda_k|) &= \sum_{k=1}^{p} \log(1 - c_k \exp[\log |\lambda_k|]) \\ &= f(\log |\lambda|) \\ &= f(S \log r) \\ &\geqq f(\log r_{i_1}, \cdots, \log r_{i_p}) \\ &= \sum_{k=1}^{p} \log(1 - c_k \exp[\log r_{i_k}]).\end{aligned} \tag{11}$$

Every r_i is less than 1 so that

$$\begin{aligned}\sum_{k=1}^{p} \log(1 - c_k \exp[\log r_{i_k}]) &\geqq \sum_{k=1}^{p} \log(1 - c_k \exp[\log r_k]) \\ &= \sum_{k=1}^{p} \log(1 - c_k r_k).\end{aligned} \tag{12}$$

Combining (11) and (12) and taking exponentials produces the lower equality in (3).

Of course, if we set $p=n$ and $c_1 = \cdots = c_n = 1$ then (3) becomes (2).

References

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