A REMARK CONCERNING QUASI-FROBENIUS RINGS

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The purpose of this note is to prove a theorem which establishes a connection between results of [1] and [2]. Recall that for any R-module M, the "dual" $M^* = \text{Hom}_R(M, R)$ has a natural structure as a module of the opposite hand from M, induced by the bi-module character of R.

THEOREM. *For any ring R, if* (a_l): *every R-operator homomorphism between minimal left ideals of R is given by a right multiplication, then* (b_l): *the dual of every simple left R-module is simple or zero. Conversely, condition* (b_l) *implies condition* (a_l), *provided that for every minimal left ideal L of R, the set $(L)^0$ of elements of R which annihilate L on the right is $\neq R$.*

The same relationship exists between the analogous conditions (a_r) *and* (b_r) *for right ideals and right modules.*

In [2, Propositions 1 and 3, pp. 204 and 206], Ikeda proved: *If A is an algebra of finite rank containing a left identity [a ring with minimum condition on left and right ideals], then A is quasi-Frobenius if and only if A satisfies* (a_l) [*both* (a_l) *and* (a_r)]. In [1, (3.4) and (4.1), pp. 349 and 350], Dieudonné proved results which are identical in statement to those of [2], cited above in italics, with conditions (a_l) and (a_r) replaced by conditions (b_l) and (b_r), respectively except that he assumed that A had an identity. Our theorem allows immediate passage from Ikeda's results to those of Dieudonné. Used together with Lemma 1 of [2, p. 204], it also allows the reverse passage.

PROOF OF THE THEOREM. ($\Rightarrow$) If S is a simple left R-module such that $S^* \neq 0$, S is isomorphic to some minimal left ideal L of R and hence $S^* \cong L^*$. Let $0 \neq f \in L^*$; then $f^{-1}: f(L) \to L$ exists and is a homomorphism between minimal left ideals of R. Consequently, there exists an element $r \in R$ such that $fr = i$ (the identity map on L). For any $f' \in L^*$, there is an $r' \in R$ such that $f' = ir'$ and hence $f' = ir' = (fr)r' = f(rr')$.

($\Leftarrow$) Let L be any minimal left ideal of R; then $L^* \neq 0$ since i belongs to L^* and hence L^* is simple. Since $(L)^0 \neq R$ there exists an element $r \in R$ such that $ir \neq 0$ and hence $iR = L^*$.

Received by the editors July 3, 1964.

[1] This work was done while the author held a NASA Fellowship at Iowa State University.

REMARKS. (1) It is easy to construct examples of rings which satisfy condition (b_l) but not condition (a_l).

(2) Any ring which satisfies condition (a_l) and contains a minimal left ideal must have $(L)^0 \neq R$ for every minimal left ideal.

The author wishes to thank Professors B. Vinograde and T. J. Head for their help in organizing this note.

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ON THE SUBGROUPS OF THE PICARD GROUP[1]

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1. **Introduction.** The Picard group Γ is important in the Theory of Automorphic Functions [3]. It consists of all linear transformations

$$w = \frac{az+b}{cz+d}, \qquad ad - bc = \pm 1 \tag{1.1}$$

with coefficients Gaussian integers. Γ is known [3] to have four generators

$$s, t, u, v \tag{1.2}$$

together with the eight defining relations

$$s^2 = u^2 = v^2 = (us^{-1})^2 = (vt^{-1})^2 = (st^{-1})^2 = (ut^{-1})^3 = (vu^{-1})^3 = 1. \tag{1.3}$$

The generators (1.2) are the transformations $w = -z$, $w = z-1$, $w = -1/z$, and $w = -z+i$ respectively.

In this paper, we seek to examine the structure of the Picard group by studying its subgroups. The modular group is a well-known subgroup. It consists of all transformations (1.1) with coefficients

Received by the editors November 19, 1964.

[1] This paper was completed under NSF grant GP-3204.